

Dynamic Bivariate Normal Copula

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Abstract Normal copula with a correlation coefficient between -1 and 1 is tail independent and so it severely underestimates extreme probabilities. By letting the correlation coefficient in a normal copula depend on the sample size, Hüsler and Reiss (1989) showed that the tail can become asymptotically dependent. In this paper, we extend this result by deriving the limit of the normalized maximum of n independent observations, where the i -th observation follows from a normal copula with its correlation coefficient being either a parametric or a nonparametric function of i/n . Furthermore, both parametric and nonparametric inference for this unknown function are studied, which can be employed to test the condition in Hüsler and Reiss (1989). A simulation study and real data analysis are presented too.

Key words Estimation; normal copula; tail dependence/independence.

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1 Introduction

Let $\{(X_i, Y_i)\}_{i=1}^n$ be independent and identically distributed random vectors with distribution function $F(x, y)$, continuous marginals F_1 and F_2 . The copula of F is defined as $F(F_1^{-1}(x), F_2^{-1}(y))$, where F_i^{-1} denotes the inverse function of F_i . Assume the copula of F follows from a normal copula $C(x, y; \rho)$, where $\rho \in [-1, 1]$ is unknown. Hence the density of $C(x, y; \rho)$ is

$$c(x, y; \rho) = \frac{1}{\sqrt{1 - \rho^2}} \exp \left(\frac{2\rho\Phi^{-1}(x)\Phi^{-1}(y) - \rho^2(\Phi^{-1}(x))^2 - \rho^2(\Phi^{-1}(y))^2}{2(1 - \rho^2)} \right) \quad (1.1)$$

for $\rho \in (-1, 1)$, where $\Phi(x)$ is the standard normal distribution function.

Normal copulas are one of most commonly used elliptical copulas, and elliptical copulas are popular in risk management due to their ease of simulation (see McNeil, Frey and Embrechts (2005)). Recently Channouf and L'Ecuyer (2012) used normal copulas to model arrive processes in a call center, Fung and Seneta (2011) showed that a bivariate normal copula is regularly varying, Meyer (2013) studied the properties of a bivariate normal copula, efficient estimation for bivariate normal copula models was studied by Klaassen

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and Wellner (1997). Although normal copulas are easy to use and have some attractive properties, a serious drawback of using a normal copula is the so-called tail asymptotic independence (see Sibuya (1960)), which under-estimates extreme probabilities in risk management.

To overcome the shortcoming of the tail asymptotic independence of a normal copula, Frick and Reiss (2013) assumed that $\rho = \rho(n)$ satisfies the so-called Hüsler-Reiss condition

$$(1 - \rho) \log n \rightarrow \lambda \in [0, \infty] \quad \text{as } n \rightarrow \infty, \quad (1.2)$$

(cf. Hüsler and Reiss (1989)) and proved that

$$\begin{aligned} & \mathbb{P} \left(n(\max_{1 \leq i \leq n} F_1(X_i) - 1) \leq x, n(\max_{1 \leq i \leq n} F_2(Y_i) - 1) \leq y \right) \\ \rightarrow & \exp \left(\Phi(\sqrt{\lambda} + \frac{\log(x/y)}{2\sqrt{\lambda}})x + \Phi(\sqrt{\lambda} + \frac{\log(y/x)}{2\sqrt{\lambda}})y \right) \end{aligned} \quad (1.3)$$

for $x < 0$ and $y < 0$ as $n \rightarrow \infty$. This is the copula version of the limit in Hüsler and Reiss (1989) for the normalized maxima of n independent random vectors with a bivariate normal distribution and its correlation coefficient satisfying (1.2). Obviously, a bivariate random vector with the above limiting distribution is dependent when $\lambda < \infty$. Extending the results in Hüsler and Reiss (1989) to elliptical triangular arrays is given in Hashorva (2005, 2006).

Since the above ρ depends on the sample size n , one may call it dynamic normal copula. Recently dynamic copulas are receiving some attention in modeling financial time series; see Benth and Kettler (2011), Mendes and de Melo (2010), Guégan and Zhang (2010), and Van den Goorbergh, Genest and Werker (2005).

In this paper, we further study the convergence in (1.3) by allowing ρ to depend on both i and n . That is, we do not assume that $(X_i, Y_i)'$ s are identically distributed. Motivated by (1.2), an obvious extension is to assume that $(1 - \rho) \log n$ is a function of i and n . As in nonparametric regression models, we assume that $(1 - \rho) \log n$ is a smoothing nonparametric or parametric function of i/n so that we can employ well-developed local polynomial techniques to estimate this function and to test whether this function is a constant, which gives a way to verify the condition imposed by Hüsler and Reiss (1989) and Frick and Reiss (2013), and indicates the observations have the same distribution. More specifically we assume that $\{(X_i, Y_i)\}_{i=1}^n$ is a sequence of independent random vectors and the copula of (X_i, Y_i) is a normal copula with correlation coefficient $\rho = 1 - m(i/n)/\log n$ for an unknown smooth function $m(x)$. After deriving the convergence for the normalized maxima of the copulas of $(X_i, Y_i)'$ s, we propose both parametric and nonparametric estimation for $m(x)$, which are based on either Kendall's tau or correlation coefficient. We also derive the asymptotic limits of the proposed estimators, which turn out to be quite nonstandard with an unusual rate of convergence. The proposed estimators can be used to determine tail dependence, which is of importance in predicting co-movement in financial markets; see McNeil, Frey and Embrechts (2005).

We organize this paper as follows. Section 2 presents the main results and statistical inference procedures. A simulation study is given in Section 3. Section 4 reports some empirical data analyses. All the proofs are given in Section 5.

2 Methodology

Throughout, suppose $\{(X_i, Y_i)\}_{i=1}^n$ are independent random vectors, X_i 's have the same continuous distribution function F_1 and Y_i 's have the same continuous distribution function F_2 . Assume the copula of (X_i, Y_i) is the normal copula $C(x, y; \rho_i)$ with density given by (1.1).

2.1 Convergence of maxima and tail coefficient

As motivated in the introduction, we extend the result (1.3) by assuming

$$\rho_i = 1 - m(i/n)/\log n \quad \text{for some nonnegative function } m(s), \quad (2.1)$$

which includes condition (1.2) as a special case.

Theorem 1. *Under condition (2.1),*

i) *if $\max_{1 \leq i \leq n} m(i/n) \rightarrow 0$, then for any $x < 0$ and $y < 0$*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(n \left(\max_{1 \leq i \leq n} F_1(X_i) - 1 \right) \leq x, n \left(\max_{1 \leq i \leq n} F_2(Y_i) - 1 \right) \leq y \right) = \exp \left(\min(x, y) \right);$$

ii) *if $\min_{1 \leq i \leq n} m(i/n) \rightarrow \infty$, then for any $x < 0$ and $y < 0$*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(n \left(\max_{1 \leq i \leq n} F_1(X_i) - 1 \right) \leq x, n \left(\max_{1 \leq i \leq n} F_2(Y_i) - 1 \right) \leq y \right) = \exp(x + y);$$

iii) *if $m(s)$ is a continuous positive function on $[0, 1]$, then for any $x < 0$ and $y < 0$*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P} \left(n \left(\max_{1 \leq i \leq n} F_1(X_i) - 1 \right) \leq x, n \left(\max_{1 \leq i \leq n} F_2(Y_i) - 1 \right) \leq y \right) \\ &= \exp \left(x \int_0^1 \Phi \left(\sqrt{m(s)} + \frac{\log(x/y)}{2\sqrt{m(s)}} \right) ds + y \int_0^1 \Phi \left(\sqrt{m(s)} + \frac{\log(y/x)}{2\sqrt{m(s)}} \right) ds \right) \\ &=: G(x, y). \end{aligned}$$

Furthermore the tail dependence function $l(x, y) = \lim_{t \rightarrow 0} t^{-1} \{1 - G(tx, ty)\}$ equals

$$-x \int_0^1 \Phi \left(\sqrt{m(s)} + \frac{\log(x/y)}{2\sqrt{m(s)}} \right) ds - y \int_0^1 \Phi \left(\sqrt{m(s)} + \frac{\log(y/x)}{2\sqrt{m(s)}} \right) ds$$

for $x < 0$ and $y < 0$, and the tail coefficient is $\lambda = l(-1, -1) = 2 \int_0^1 \Phi(\sqrt{m(s)}) ds$.

2.2 Parametric inference

Here we consider statistical inference for fitting a parametric form to the unknown function $m(s)$. First, we consider the family $m(s) = \alpha + \beta s^\gamma$, where $\alpha > 0, \beta \neq 0, \gamma > 0$. Note that when $\beta = 0$, γ can not be identified. Also when $\gamma = 0$, α and β can not be distinguished.

It follows from Theorem 5.36 of McNeil, Frey and Embrechts (2005) that

$$\mathbb{E} \left(\operatorname{sgn}((U_i - \tilde{U}_i)(V_i - \tilde{V}_i)) \right) = \frac{2}{\pi} \arcsin(\rho_i), \quad (2.2)$$

where $(\tilde{U}_i, \tilde{V}_i)$ is an independent copy of (U_i, V_i) , and

$$\mathbb{E} \left((U_i - \frac{1}{2})(V_i - \frac{1}{2}) \right) = \frac{1}{2\pi} \arcsin(\frac{\rho_i}{2}). \quad (2.3)$$

Also we have

$$\mathbb{E} \left(\Phi^-(F_1(X_i))\Phi^-(F_2(Y_i)) \right) = \rho_i. \quad (2.4)$$

Therefore, one can employ the standard least squares estimate based on one of the above equations.

Since $(U_i, V_i)'$ s are not identically distributed, we do not have an independent copy of (U_i, V_i) , which prevents us from using (2.2). Hence we propose to use either (2.3) or (2.4) to construct the least squares estimator, which results in

$$(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) = \arg \min_{(\alpha, \beta, \gamma)} \sum_{i=1}^n \left((\hat{F}_1(X_i) - \frac{1}{2})(\hat{F}_2(Y_i) - \frac{1}{2}) - \frac{1}{2\pi} \arcsin(\frac{1 - (\alpha + \beta(i/n)^\gamma)/\log n}{2}) \right)^2$$

or

$$(\hat{\alpha}^*, \hat{\beta}^*, \hat{\gamma}^*) = \arg \min_{(\alpha, \beta, \gamma)} \sum_{i=1}^n \left(\Phi^-(\hat{F}_1(X_i))\Phi^-(\hat{F}_2(Y_i)) - 1 + \frac{\alpha + \beta(i/n)^\gamma}{\log n} \right)^2,$$

where $\hat{F}_1(x) = \frac{1}{n+1} \sum_{i=1}^n I(X_i \leq x)$ and $\hat{F}_2(y) = \frac{1}{n+1} \sum_{i=1}^n I(Y_i \leq y)$. Alternatively we define $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ to be the solution to the following score equations

$$\begin{cases} l_{n1}(\alpha, \beta, \gamma) := \sum_{i=1}^n \left((\hat{F}_1(X_i) - \frac{1}{2})(\hat{F}_2(Y_i) - \frac{1}{2}) - \frac{1}{2\pi} \arcsin(\frac{\rho_i}{2}) \right) = 0, \\ l_{n2}(\alpha, \beta, \gamma) := \sum_{i=1}^n \left((\hat{F}_1(X_i) - \frac{1}{2})(\hat{F}_2(Y_i) - \frac{1}{2}) - \frac{1}{2\pi} \arcsin(\frac{\rho_i}{2}) \right) \left(\frac{i}{n} \right)^\gamma = 0, \\ l_{n3}(\alpha, \beta, \gamma) := \sum_{i=1}^n \left((\hat{F}_1(X_i) - \frac{1}{2})(\hat{F}_2(Y_i) - \frac{1}{2}) - \frac{1}{2\pi} \arcsin(\frac{\rho_i}{2}) \right) \left(\frac{i}{n} \right)^\gamma \log\left(\frac{i}{n}\right) = 0 \end{cases} \quad (2.5)$$

and $(\hat{\alpha}^*, \hat{\beta}^*, \hat{\gamma}^*)$ to be the solution to the following score equations

$$\begin{cases} l_{n1}^*(\alpha, \beta, \gamma) := \sum_{i=1}^n \left(\Phi^-(\hat{F}_1(X_i))\Phi^-(\hat{F}_2(Y_i)) - 1 + \frac{\alpha + \beta(i/n)^\gamma}{\log n} \right) = 0, \\ l_{n2}^*(\alpha, \beta, \gamma) := \sum_{i=1}^n \left(\Phi^-(\hat{F}_1(X_i))\Phi^-(\hat{F}_2(Y_i)) - 1 + \frac{\alpha + \beta(i/n)^\gamma}{\log n} \right) \left(\frac{i}{n} \right)^\gamma = 0, \\ l_{n3}^*(\alpha, \beta, \gamma) := \sum_{i=1}^n \left(\Phi^-(\hat{F}_1(X_i))\Phi^-(\hat{F}_2(Y_i)) - 1 + \frac{\alpha + \beta(i/n)^\gamma}{\log n} \right) \left(\frac{i}{n} \right)^\gamma \log\left(\frac{i}{n}\right) = 0. \end{cases} \quad (2.6)$$

Note that we skip the term of $\frac{d}{d\rho_i} \arcsin(\rho_i/2)$ in (2.5), which goes to a constant uniformly in i since $\rho_i \rightarrow 1$ uniformly in i .

The following theorems give the asymptotic normality of the proposed estimators.

Theorem 2. Suppose (2.1) holds with $m(s) = \alpha + \beta s^\gamma$ for some $\alpha > 0, \beta \neq 0, \gamma > 0$. Then we have

$$\begin{pmatrix} \frac{\sqrt{n}}{(\log n)^{3/4}} & 0 & 0 \\ 0 & \frac{\sqrt{n}}{\log n} & 0 \\ 0 & 0 & \frac{\sqrt{n}}{\log n} \end{pmatrix} \hat{\Delta} \begin{pmatrix} \hat{\alpha} - \alpha \\ \hat{\beta} - \beta \\ \hat{\gamma} - \gamma \end{pmatrix} \xrightarrow{d} N(0, \Sigma) \quad (2.7)$$

and

$$\left(\frac{\sqrt{n}}{\log n}(\hat{\alpha} - \alpha), \frac{\sqrt{n}}{\log n}(\hat{\beta} - \beta), \frac{\sqrt{n}}{\log n}(\hat{\gamma} - \gamma) \right)^T \xrightarrow{d} N\left(0, \Delta^{-1} \Sigma_0 (\Delta^{-1})^T\right), \quad (2.8)$$

where

$$\Sigma = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & \sigma_{23} & \sigma_{33} \end{pmatrix}, \quad \Sigma_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & \sigma_{23} & \sigma_{33} \end{pmatrix},$$

$$\Delta = \frac{\sqrt{3}}{6\pi} \begin{pmatrix} 1 & \frac{1}{1+\gamma} & -\frac{\beta}{(1+\gamma)^2} \\ \frac{1}{1+\gamma} & \frac{1}{1+2\gamma} & -\frac{\beta}{(1+2\gamma)^2} \\ -\frac{1}{(1+\gamma)^2} & -\frac{1}{(1+2\gamma)^2} & \frac{2\beta}{(1+2\gamma)^3} \end{pmatrix}, \quad \hat{\Delta} = \frac{\sqrt{3}}{6\pi} \begin{pmatrix} 1 & \frac{1}{1+\hat{\gamma}} & -\frac{\hat{\beta}}{(1+\hat{\gamma})^2} \\ \frac{1}{1+\hat{\gamma}} & \frac{1}{1+2\hat{\gamma}} & -\frac{\hat{\beta}}{(1+2\hat{\gamma})^2} \\ -\frac{1}{(1+\hat{\gamma})^2} & -\frac{1}{(1+2\hat{\gamma})^2} & \frac{2\hat{\beta}}{(1+2\hat{\gamma})^3} \end{pmatrix}.$$

Theorem 3. Suppose (2.1) holds with $m(s) = \alpha + \beta s^\gamma$ for some $\alpha > 0, \beta \neq 0, \gamma > 0$. Then we have

$$\begin{pmatrix} \sqrt{n} & 0 & 0 \\ 0 & \frac{\sqrt{n}}{\log n} & 0 \\ 0 & 0 & \frac{\sqrt{n}}{\log n} \end{pmatrix} \hat{\Delta}^* \begin{pmatrix} \hat{\alpha}^* - \alpha \\ \hat{\beta}^* - \beta \\ \hat{\gamma}^* - \gamma \end{pmatrix} \xrightarrow{d} N(0, \Sigma^*) \quad (2.9)$$

and

$$\left(\frac{\sqrt{n}}{\log n} (\hat{\alpha}^* - \alpha), \frac{\sqrt{n}}{\log n} (\hat{\beta}^* - \beta), \frac{\sqrt{n}}{\log n} (\hat{\gamma}^* - \gamma) \right)^T \xrightarrow{d} N(0, (\Delta^*)^{-1} \Sigma_0^* (\Delta^{*T})^{-1}), \quad (2.10)$$

where

$$\Sigma^* = \begin{pmatrix} \sigma_{11}^* & 0 & 0 \\ 0 & \sigma_{22}^* & \sigma_{23}^* \\ 0 & \sigma_{23}^* & \sigma_{33}^* \end{pmatrix}, \quad \Sigma_0^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22}^* & \sigma_{23}^* \\ 0 & \sigma_{23}^* & \sigma_{33}^* \end{pmatrix},$$

$$\begin{cases} \sigma_{11}^* = 4(\alpha + \frac{\beta}{1+\gamma})^2 + 2\beta^2(\frac{1}{1+2\gamma} - \frac{1}{(1+\gamma)^2}), & \sigma_{22}^* = \frac{2}{1+2\gamma} - \frac{2}{(1+\gamma)^2}, \\ \sigma_{33}^* = \frac{4}{(1+2\gamma)^3} - \frac{2}{(1+\gamma)^4}, & \sigma_{23}^* = -\frac{2}{(1+2\gamma)^2} + \frac{2}{(1+\gamma)^3}, \end{cases}$$

$\Delta^* = 2\sqrt{3}\pi\Delta$ and $\hat{\Delta}^* = 2\sqrt{3}\pi\hat{\Delta}$, where Δ and $\hat{\Delta}$ are given in Theorem 2.

Remark 1. Since $\sigma_{22} < \sigma_{22}^*/(12\pi^2)$ and $\sigma_{33} < \sigma_{33}^*/(12\pi^2)$, $\hat{\beta}$ and $\hat{\gamma}$ have a smaller asymptotic variance than $\hat{\beta}^*$ and $\hat{\gamma}^*$, respectively, while the comparison for the asymptotic variances of $\hat{\alpha}$ and $\hat{\alpha}^*$ is unclear since both $\frac{\sqrt{n}}{\log n}(\hat{\alpha} - \alpha)$ and $\frac{\sqrt{n}}{\log n}(\hat{\alpha}^* - \alpha)$ converge in distribution to zero. On the other hand, if one is interested in estimating $\Delta(\alpha, \beta, \gamma)^T$, then the estimator for the first element based on $(\hat{\alpha}^*, \hat{\beta}^*, \hat{\gamma}^*)^T$ has a faster rate of convergence than the corresponding estimator based on $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})^T$, but the estimators for the second and third elements based on $(\hat{\alpha}^*, \hat{\beta}^*, \hat{\gamma}^*)^T$ have a larger asymptotic variance than those based on $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})^T$. In spite of these theoretical comparisons, the simulation study below does prefer the estimation procedure based on equation (2.3) when the mean squared error is concerned. For testing $(\alpha, \beta, \gamma)^T = (\alpha_0, \beta_0, \gamma_0)^T$, one should employ the well-known Hotelling T^2 test statistic based on either (2.7) or (2.9) because the limit in both (2.8) and (2.10) is degenerate.

Another interesting parametric form for $m(s)$ is polynomial. Here we consider $m(s) = \alpha + \beta s$. In this case, when $\beta = 0$, $m(s)$ becomes constant, which means that the observations $(X_1, Y_1), \dots, (X_n, Y_n)$ are independent and identically distributed random vectors.

Theorem 4. Suppose (2.1) holds with $m(s) = \alpha + \beta s$ for some $\alpha > 0, \beta \in R$. Then we have

$$\left(\frac{\sqrt{n}}{(\log n)^{3/4}} (\hat{\alpha} + \frac{\hat{\beta}}{2} - \alpha - \frac{\beta}{2}), \frac{\sqrt{n}}{\log n} (\frac{\hat{\alpha}}{2} + \frac{\hat{\beta}}{3} - \frac{\alpha}{2} - \frac{\beta}{3}) \right)^T \xrightarrow{d} N(0, \tilde{\Sigma}),$$

where $\tilde{\Sigma} = 12\pi^2(\tilde{\sigma}_{ij})$ is a symmetric matrix with

$$\tilde{\sigma}_{11} = \frac{2\sqrt{2}}{3\beta}((\alpha + \beta)^{3/2} - \alpha^{3/2}) \int_0^1 (u - \frac{1}{2})^2 \phi(\Phi^-(u)) du, \quad \tilde{\sigma}_{22} = \frac{1}{2160}, \quad \tilde{\sigma}_{12} = 0.$$

Theorem 5. Suppose (2.1) holds with $m(s) = \alpha + \beta s$ for some $\alpha > 0, \beta \in R$. Then we have

$$\left(\sqrt{n}(\hat{\alpha}^* + \frac{\hat{\beta}^*}{2} - \alpha - \frac{\beta}{2}), \frac{\sqrt{n}}{\log n}(\frac{\hat{\alpha}^*}{2} + \frac{\hat{\beta}^*}{3} - \frac{\alpha}{2} - \frac{\beta}{3}) \right)^T \xrightarrow{d} N(0, \tilde{\Sigma}^*),$$

where $\tilde{\Sigma}^* = (\tilde{\sigma}_{ij}^*)$ is a symmetric matrix with

$$\tilde{\sigma}_{11}^* = 4\alpha^2 + 4\alpha\beta + \frac{7\beta^2}{6}, \quad \tilde{\sigma}_{22}^* = \frac{1}{6}, \quad \tilde{\sigma}_{12}^* = 0.$$

Remark 2. When (2.1) holds with $m(s) = \alpha$, we can show the rate of convergence for $\hat{\alpha}^*$ is faster than the rate of convergence for $\hat{\alpha}$. That is, the estimator based on (2.4) is preferred to that based on (2.3). However, the simulation study below prefers the estimation procedure based on equation (2.3) when the mean squared error is used as a criterion.

2.3 Nonparametric inference

First we use (2.3) to estimate the smooth function $Q(s) = \frac{1}{2\pi} \arcsin(\frac{1-m(s)/\log n}{2})$ nonparametrically. Especially we consider the local linear estimator $\hat{Q}(s)$ defined as

$$(\hat{Q}(s), \hat{b}) = \arg \min_{a,b} \sum_{i=1}^n \left((\hat{F}_1(X_i) - \frac{1}{2})(\hat{F}_2(Y_i) - \frac{1}{2}) - a - b(s - i/n) \right) k\left(\frac{s - i/n}{h}\right),$$

where k is a kernel function and $h = h(n) \rightarrow 0$ is a bandwidth. That is,

$$\hat{Q}(s) = \frac{\sum_{j=1}^n w_j (\hat{F}_1(X_j) - \frac{1}{2})(\hat{F}_2(Y_j) - \frac{1}{2})}{\sum_{j=1}^n w_j},$$

where $w_j = k(\frac{s-j/n}{h})[s_{n,2} - (s - j/n)s_{n,1}]$, $s_{n,l} = \sum_{j=1}^n k(\frac{s-j/n}{h})(s - j/n)^l$. We refer to Fan and Gijbels [3] for details. Therefore we can estimate $m(s)$ non parametrically by

$$\hat{m}(s) = \left(1 - 2 \sin(2\pi \hat{Q}(s))\right) \log n.$$

Theorem 6. Assume $k(s)$ is symmetric with support $[-1, 1]$. For a given $s \in (0, 1)$, assume $m''(t)$ is continuous at s , $h = h(n) \rightarrow 0$ and $\frac{h^2 \sqrt{n} h}{\log n} \rightarrow \lambda$ as $n \rightarrow \infty$. Then as $n \rightarrow \infty$ we have

$$\frac{\sqrt{n} h}{\log n} \left(\hat{m}(s) - m(s) \right) \xrightarrow{d} N \left(\frac{1}{2} \lambda m''(s) \int_{-1}^1 t^2 k(t) dt, \frac{\pi^2}{15} \int_{-1}^1 k^2(t) dt \right).$$

Second we use (2.4) to estimate the smooth function $m(s)$ nonparametrically by considering the local linear estimator

$$(\hat{m}^*(s), \hat{b}) = \arg \min_{a,b} \sum_{i=1}^n \left(\Phi^-(\hat{F}_1(X_i)) \Phi^-(\hat{F}_2(Y_i)) - 1 + \frac{a}{\log n} + \frac{b}{\log n} (s - i/n) \right) k\left(\frac{s - i/n}{h}\right),$$

i.e.,

$$\hat{m}^*(s) = - \frac{\sum_{j=1}^n w_j (\Phi^-(\hat{F}_1(X_j)) \Phi^-(\hat{F}_2(Y_j)) - 1) \log n}{\sum_{j=1}^n w_j}.$$

Theorem 7. Assume $k(s)$ is symmetric with support $[-1, 1]$. For a given $s \in (0, 1)$, assume $m''(t)$ is continuous at s , $h = h(n) \rightarrow 0$ and $\frac{h^2 \sqrt{nh}}{\log n} \rightarrow \lambda$ as $n \rightarrow \infty$. Then as $n \rightarrow \infty$ we have

$$\frac{\sqrt{nh}}{\log n} (\hat{m}^*(s) - m(s)) \xrightarrow{d} N \left(\frac{1}{2} \lambda m''(s) \int_{-1}^1 t^2 k(t) dt, 2 \int_{-1}^1 k^2(t) dt \right).$$

Remark 3. It follows from Theorems 6 and 7 that both $\hat{m}^*(s)$ and $\hat{m}(s)$ have the same asymptotic bias, but $\hat{m}(s)$ has a smaller asymptotic variance than $\hat{m}^*(s)$. Hence, unlike parametric estimation, nonparametric estimation based on (2.3) is always preferred.

Remark 4. By minimizing the asymptotic mean squared error, the optimal choices of h for $\hat{m}(s)$ and $\hat{m}^*(s)$ are

$$h_0 = \left(\frac{\log^2 n}{n} \right)^{1/5} \left(\frac{\pi^2 \int_{-1}^1 k^2(t) dt}{15(m''(s) \int_{-1}^1 t^2 k(t) dt)^2} \right)^{1/5}$$

and

$$h_0^* = \left(\frac{\log^2 n}{n} \right)^{1/5} \left(\frac{2 \int_{-1}^1 k^2(t) dt}{(m''(s) \int_{-1}^1 t^2 k(t) dt)^2} \right)^{1/5},$$

respectively, which are different from the standard optimal order $n^{-1/5}$ in the bandwidth choice of nonparametric regression estimation and nonparametric density estimation. Data driven method for choosing the above h_0 and h_0^* can be obtained via estimating $m''(s)$. A future research is to investigate the possibility of using cross-validation method to choose the optimal bandwidth.

Remark 5. It is straightforward to construct both parametric and nonparametric estimation for the tail dependence function and the tail coefficient given in Theorem 1 and to derive the corresponding asymptotic results by using Theorems 2-7.

3 Simulation

In this section we examine the finite sample performance of the proposed estimators by drawing independent $(X_1, Y_1), \dots, (X_n, Y_n)$ with (X_i, Y_i) following the normal copula with correlation coefficient $\rho = 1 - m(i/n)/\log n$. We consider $n = 300$ or 1000 or 3000 , and repeat 1000 times.

First we consider $m(s) = \alpha$ with $\alpha = 1$ or 10 , and calculate the average, sample variance and mean squared error for both $\hat{\alpha}$ and $\hat{\alpha}^*$. Table 1 below shows that $\hat{\alpha}^*$ has a smaller variance than $\hat{\alpha}$, which confirms the argument mentioned in Remark 2 that estimator $\hat{\alpha}^*$ has a faster rate of convergence than $\hat{\alpha}$. We also observe from Table 1 that i) $\hat{\alpha}^*$ has a larger bias and a larger mean squared error than $\hat{\alpha}$ except the case of $\alpha = 10$ and $n = 3000$; ii) the variance and mean squared error of both $\hat{\alpha}$ and $\hat{\alpha}^*$ become larger when α increases; iii) the accuracy for both estimators improves as n becomes larger. In conclusion, $\hat{\alpha}$ has an overall better finite sample behavior in terms of mean squared error than $\hat{\alpha}^*$ although its asymptotic variance is larger theoretically and empirically.

Next we consider the case of $m(s) = \alpha + \beta s$. In Table 2 we report the average, sample variance and mean squared error for estimators $(\hat{\alpha}, \hat{\beta})$, $(\hat{\alpha}^*, \hat{\beta}^*)$, $(\hat{\alpha} + \frac{\hat{\beta}}{2}, \frac{\hat{\alpha}}{2} + \frac{\hat{\beta}}{3})$ and $(\hat{\alpha}^* + \frac{\hat{\beta}^*}{2}, \frac{\hat{\alpha}^*}{2} + \frac{\hat{\beta}^*}{3})$. As we see, estimators $(\hat{\alpha}, \hat{\beta})$ have a smaller variance than $(\hat{\alpha}^*, \hat{\beta}^*)$, but $\hat{\alpha}^* + \frac{\hat{\beta}^*}{2}$ has a smaller variance than $\hat{\alpha} + \frac{\hat{\beta}}{2}$, which

Table 1: Estimators for the case of $m(s) = \alpha$.

	$\alpha = 1$	$\alpha = 10$	$\alpha = 1$	$\alpha = 10$	$\alpha = 1$	$\alpha = 10$
	$n = 300$	$n = 300$	$n = 1000$	$n = 1000$	$n = 3000$	$n = 3000$
$E(\hat{\alpha})$	1.0365	9.9660	1.0145	9.9836	1.0065	9.9976
$V(\hat{\alpha})$	0.0144	0.0249	0.0045	0.0384	0.0016	0.0218
$MSE(\hat{\alpha})$	0.0157	0.0203	0.0047	0.0387	0.0016	0.0218
$E(\hat{\alpha}^*)$	1.1690	9.8440	1.0788	9.9476	1.0368	9.9914
$V(\hat{\alpha}^*)$	0.0109	0.0191	0.0034	0.0322	0.0012	0.0193
$MSE(\hat{\alpha}^*)$	0.0395	0.0434	0.0096	0.0349	0.0026	0.0194

is supported by Theorems 4 and 5 that $\hat{\alpha}^* + \frac{\hat{\beta}^*}{2}$ has a faster rate of convergence than $\hat{\alpha} + \frac{\hat{\beta}}{2}$. As n becomes larger, the accuracy of all estimators improves. Since $\hat{\alpha}$ and $\hat{\beta}$ have a smaller mean squared error than $\hat{\alpha}^*$ and $\hat{\beta}^*$, respectively, we prefer the estimation procedure based on equation (2.3) to that based on equation (2.4).

Finally we consider the case of $m(s) = \alpha + \beta s^\gamma$. Given results in Tables 1 and 2, we only consider the estimators derived from equation (2.3) with the large sample size $n = 3000$. Table 3 shows that all estimators have a rather large variance for $\gamma = 1$, and the variance of $\hat{\gamma}$ is still quite big even when $\gamma = 0.5$, which means estimating the shape parameter γ is very challenging as usually.

4 Data Analysis

In this section we apply the proposed nonparametric estimators to two real data sets: Danish fire loss and log-returns of exchange rates; see Figure 1.

This first data set is the nonzero losses to building and content in the Danish fire insurance claims, which comprises 2167 fire losses over the period 1980 to 1990. The second data set is the log-returns of the exchange rates between Euro and US dollar and those between British pound and US dollar from January 3, 2000 till December 19, 2007.

We calculate both $\hat{m}(s)$ and $\hat{m}^*(s)$ for $s = 0.1, 0.11, 0.12, \dots, 0.9$ by using Epanechnikov kernel $k(x) = \frac{3}{4}(1 - x^2)I(|x| \leq 1)$ and the bandwidth $h = d\{\log^2(n)/n\}^{1/5}$ with $d = 0.2, 0.3, 0.4, 0.5$. From Figures 2 and 3, we observe that $\hat{m}(s)$ and $\hat{m}^*(s)$ have a quite similar pattern for the second data set, but seem having a different pattern for the first data set when a large bandwidth is employed. To further investigate this issue, we plot the difference of $\hat{m}(s) - \hat{m}^*(s)$ in Figure 4 for the above h with $d = 0.2, 0.3, 0.4, 0.5$, which indeed shows the differences for $d = 0.4$ are quite similar to those for $d = 0.5$. Nevertheless, Remark 3 says that one should prefer $\hat{m}(s)$ to $\hat{m}^*(s)$. The non-constant $m(s)$ function indicates observations are not identically distributed.

Table 2: Estimators for the case of $m(s) = \alpha + \beta s$ with $\alpha = 1$.

	$\beta = 1$	$\beta = 0$	$\beta = 1$	$\beta = 0$	$\beta = 1$	$\beta = 0$
	$n = 300$	$n = 300$	$n = 1000$	$n = 1000$	$n = 3000$	$n = 3000$
$E(\hat{\alpha})$	1.0289	1.0270	1.0350	1.0266	1.0019	1.0000
$V(\hat{\alpha})$	0.2901	0.3230	0.1245	0.1327	0.0453	0.0528
$MSE(\hat{\alpha})$	0.2909	0.3237	0.1257	0.1334	0.0453	0.0528
$E(\hat{\beta})$	1.0345	0.0486	0.9612	-0.0240	1.0022	0.0111
$V(\hat{\beta})$	1.1597	1.2678	0.5080	0.5095	0.1833	0.2082
$MSE(\hat{\beta})$	1.1609	1.2702	0.5095	0.5101	0.1833	0.2083
$E(\hat{\alpha} + \frac{\hat{\beta}}{2})$	1.5461	1.0513	1.5030	1.0146	1.5030	1.0055
$V(\hat{\alpha} + \frac{\hat{\beta}}{2})$	0.0270	0.0150	0.0097	0.0047	0.0032	0.0015
$MSE(\hat{\alpha} + \frac{\hat{\beta}}{2})$	0.0291	0.0176	0.0097	0.0049	0.0041	0.0015
$E(\frac{\hat{\alpha}}{2} + \frac{\hat{\beta}}{3})$	0.8593	0.5297	0.8379	0.5053	0.8350	0.5037
$V(\frac{\hat{\alpha}}{2} + \frac{\hat{\beta}}{3})$	0.0170	0.0131	0.0070	0.0047	0.0024	0.0019
$MSE(\frac{\hat{\alpha}}{2} + \frac{\hat{\beta}}{3})$	0.0177	0.0140	0.0070	0.0047	0.0024	0.0019
$E(\hat{\alpha}^*)$	1.1654	1.1557	1.1155	1.0880	1.0349	1.0292
$V(\hat{\alpha}^*)$	0.4281	0.4482	0.2303	0.2246	0.0934	0.1121
$MSE(\hat{\alpha}^*)$	0.4555	0.4724	0.2436	0.2323	0.0946	0.1130
$E(\hat{\beta}^*)$	0.9802	0.0338	0.9188	-0.0177	0.9931	0.0147
$V(\hat{\beta}^*)$	1.6477	1.7563	0.9138	0.8838	0.3686	0.4504
$MSE(\hat{\beta}^*)$	1.6481	1.7575	0.9204	0.8841	0.3686	0.4506
$E(\hat{\alpha}^* + \frac{\hat{\beta}^*}{2})$	1.6555	1.1726	1.5749	1.0792	1.5315	1.0365
$V(\hat{\alpha}^* + \frac{\hat{\beta}^*}{2})$	0.0221	0.0112	0.0076	0.0037	0.0025	0.0012
$MSE(\hat{\alpha}^* + \frac{\hat{\beta}^*}{2})$	0.0463	0.0410	0.0132	0.0100	0.0035	0.0025
$E(\frac{\hat{\alpha}^*}{2} + \frac{\hat{\beta}^*}{3})$	0.9094	0.5891	0.8640	0.5381	0.8485	0.5195
$V(\frac{\hat{\alpha}^*}{2} + \frac{\hat{\beta}^*}{3})$	0.0174	0.0152	0.0087	0.0071	0.0033	0.0036
$MSE(\frac{\hat{\alpha}^*}{2} + \frac{\hat{\beta}^*}{3})$	0.0232	0.0231	0.0096	0.0086	0.0035	0.0040

Table 3: Estimators for the case of $m(s) = \alpha + \beta s^\gamma$ with $\alpha = \beta = 1$.

	$E(\hat{\alpha})$	$V(\hat{\alpha})$	$MSE(\hat{\alpha})$	$E(\hat{\beta})$	$V(\hat{\beta})$	$MSE(\hat{\beta})$	$E(\hat{\gamma})$	$V(\hat{\gamma})$	$MSE(\hat{\gamma})$
$\gamma = 0.5$	0.8631	0.2050	0.2237	1.2268	0.2840	0.3354	0.9787	8.0693	8.2985
$\gamma = 1$	0.9964	15.8532	15.8532	1.1412	16.2379	16.2578	1.7859	11.1177	11.7353

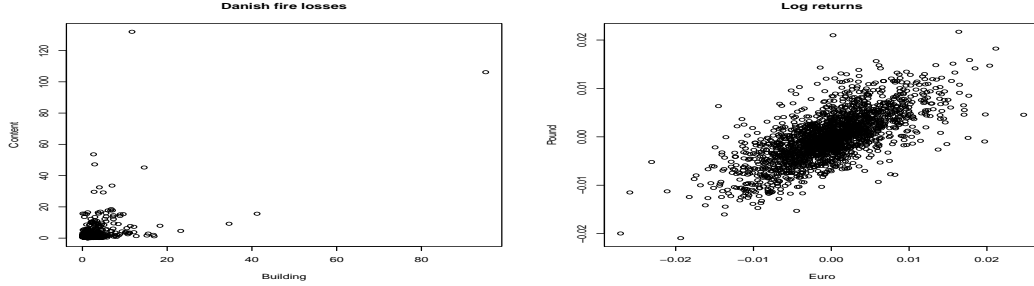


Figure 1: Left panel: Danish fire loss with 2167 fire losses over the period 1980 to 1990. Right panel: log-returns of exchange rates between Euro and US dollar and those between British pound and US dollar from January 3, 2000 till December 19, 2007.

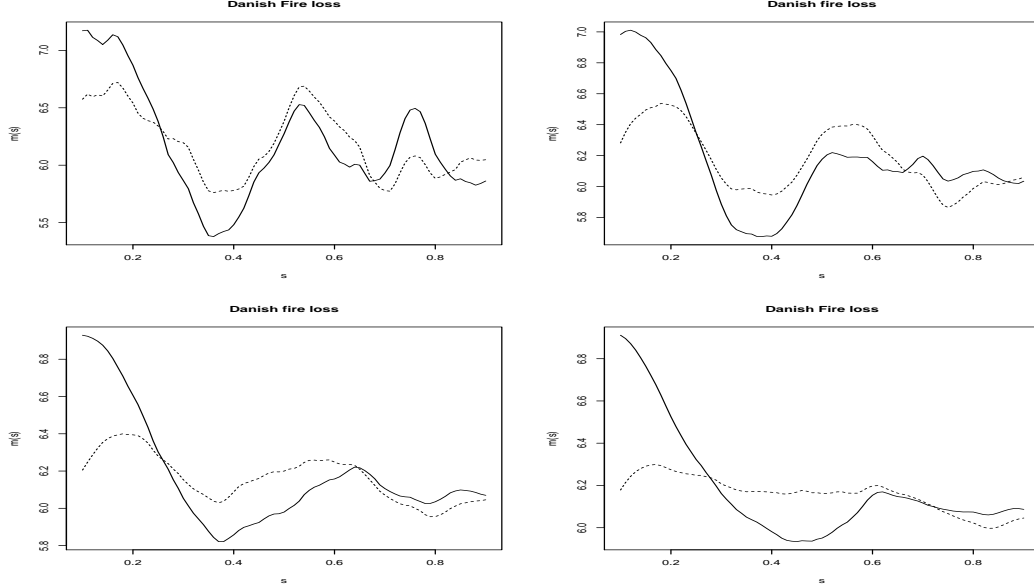


Figure 2: Danish fire losses. Solid line and dotted line represent $\hat{m}(s)$ and $\hat{m}^*(s)$, respectively. Bandwidth $h = d\{\log^2(n)/n\}^{1/5}$ with $d = 0.2, 0.3, 0.4, 0.5$ is employed in the upper left, upper right, lower left, lower right panels, respectively.

5 Proofs

Proof of Theorem 1. We focus on the proof of case iii) since the other two cases can be verified easily.

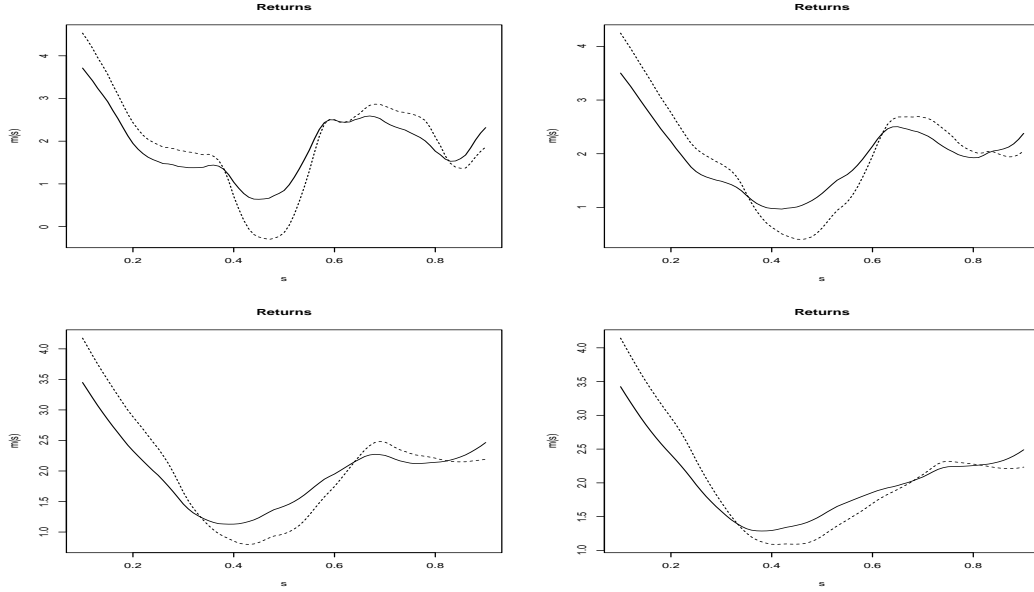


Figure 3: Exchange rates. Solid line and dotted line represent $\hat{m}(s)$ and $\hat{m}^*(s)$, respectively. Bandwidth $h = d\{\log^2(n)/n\}^{1/5}$ with $d = 0.2, 0.3, 0.4, 0.5$ is employed in the upper left, upper right, lower left, lower right panels, respectively.

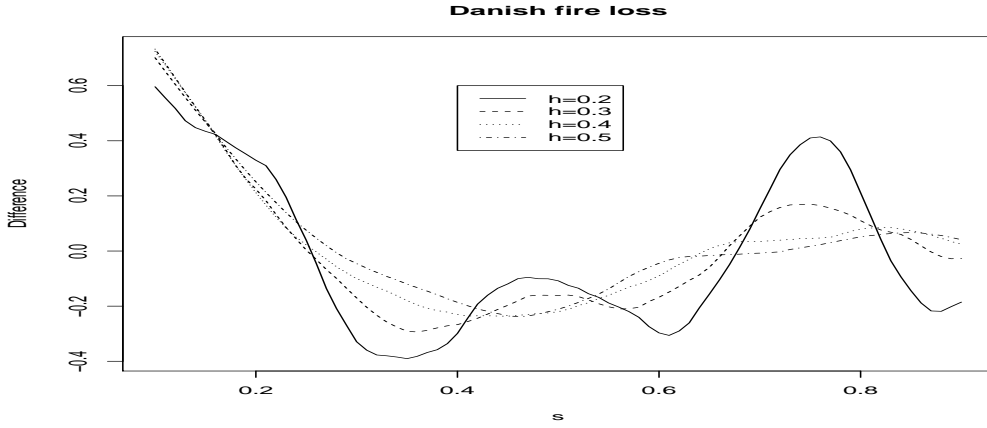


Figure 4: Danish fire losses. Differences of $\hat{m}(s) - \hat{m}^*(s)$ are plotted for bandwidth $h = d\{\log^2(n)/n\}^{1/5}$ with $d = 0.2, 0.3, 0.4, 0.5$.

For any $\epsilon > 0$ such that $\epsilon < -x$, write

$$\begin{aligned}
& 1 - \mathbb{P}(\Phi^-(F_1(X_i)) \leq \Phi^-(1 + \frac{x}{n}), \Phi^-(F_2(Y_i)) \leq \Phi^-(1 + \frac{y}{n})) \\
&= -\frac{x}{n} - \frac{y}{n} - \mathbb{P}\left(\Phi^-(F_1(X_i)) > \Phi^-(1 + \frac{x}{n}), \Phi^-(F_2(Y_i)) > \Phi^-(1 + \frac{y}{n})\right) \\
&= -\frac{x}{n} - \frac{y}{n} - \int_{\Phi^-(1+x/n)}^{\infty} \left(1 - \Phi\left(\frac{\Phi^-(1+y/n) - \rho_i s}{\sqrt{1-\rho_i^2}}\right)\right) d\Phi(s) \\
&= -\frac{x}{n} - \frac{y}{n} - n^{-1} \int_x^0 \left(1 - \Phi\left(\frac{\Phi^-(1+y/n) - \rho_i \Phi^-(1+s/n)}{\sqrt{1-\rho_i^2}}\right)\right) ds \\
&= -\frac{y}{n} - n^{-1} \int_{-\epsilon}^x \Phi\left(\frac{\Phi^-(1+y/n) - \rho_i \Phi^-(1+s/n)}{\sqrt{1-\rho_i^2}}\right) ds - n^{-1} \int_0^{-\epsilon} \Phi\left(\frac{\Phi^-(1+y/n) - \rho_i \Phi^-(1+s/n)}{\sqrt{1-\rho_i^2}}\right) ds.
\end{aligned} \tag{5.1}$$

For fixed $x < 0$ and $y < 0$, we have

$$\Phi^-(1 + y/n) = \sqrt{2 \log n} - \frac{\log(-y)}{\sqrt{2 \log n}} - \frac{\log \log n + \log(4\pi)}{2\sqrt{2 \log n}} + o\left(\frac{\log \log n}{\sqrt{\log n}}\right)$$

and

$$\Phi^-(1 + s/n) = \sqrt{2 \log n} - \frac{\log(-s)}{\sqrt{2 \log n}} - \frac{\log \log n + \log(4\pi)}{2\sqrt{2 \log n}} + o\left(\frac{\log \log n}{\sqrt{\log n}}\right)$$

uniformly in $s \in [x, -\epsilon]$, which implies that

$$\begin{aligned} & \frac{\Phi^-(1+y/n) - \rho_i \Phi^-(1+s/n)}{\sqrt{1-\rho_i^2}} \\ = & \frac{\sqrt{2 \log n} \sqrt{1-\rho_i}}{\sqrt{1+\rho_i}} - \frac{\log(-y)}{\sqrt{2 \log n} \sqrt{1-\rho_i} \sqrt{1+\rho_i}} + \frac{\rho_i \log(-s)}{\sqrt{2 \log n} \sqrt{1-\rho_i} \sqrt{1+\rho_i}} - \frac{\log \log n + \log(4\pi)}{2\sqrt{2 \log n}} \frac{\sqrt{1-\rho_i}}{\sqrt{1+\rho_i}} + o\left(\frac{\log \log n}{\sqrt{\log n}}\right) \\ = & \frac{\sqrt{2m(i/n)}}{\sqrt{2-m(i/n)/\log n}} - \frac{\log(-y)}{\sqrt{2m(i/n)} \sqrt{2-m(i/n)/\log n}} + \frac{(1-m(i/n)/\log n) \log(-s)}{\sqrt{2m(i/n)} \sqrt{2-m(i/n)/\log n}} + o\left(\frac{\log \log n}{\sqrt{\log n}}\right) \end{aligned} \quad (5.2)$$

uniformly for $s \in [x, -\epsilon]$, where $x < 0$ and $y < 0$ are fixed and $\epsilon \in (0, -x)$ is any given constant.

Since $m(s)$ is a continuous positive function, it follows from (5.2) that

$$\begin{aligned} & n^{-1} \int_{-\epsilon}^x \Phi\left(\frac{\Phi^-(1+y/n) - \rho_i \Phi^-(1+s/n)}{\sqrt{1-\rho_i^2}}\right) ds \\ = & n^{-1} \left(\int_{-\epsilon}^x \Phi\left(\sqrt{m(i/n)} - \frac{\log(-y)}{2\sqrt{m(i/n)}} + \frac{\log(-s)}{2\sqrt{m(i/n)}}\right) ds \right) (1 + o(1)) \\ = & \left(n^{-1} x \Phi\left(\sqrt{m(i/n)} + \frac{\log(x/y)}{2\sqrt{m(i/n)}}\right) + n^{-1} \epsilon \Phi\left(\sqrt{m(i/n)} + \frac{\log(-\epsilon/y)}{2\sqrt{m(i/n)}}\right) \right. \\ & \left. - n^{-1} \frac{1}{2\sqrt{m(i/n)}} \int_{-\epsilon}^x \phi\left(\sqrt{m(i/n)} - \frac{\log(-y)}{2\sqrt{m(i/n)}} + \frac{\log(-s)}{2\sqrt{m(i/n)}}\right) ds \right) (1 + o(1)) \\ = & \left(n^{-1} x \Phi\left(\sqrt{m(i/n)} + \frac{\log(x/y)}{2\sqrt{m(i/n)}}\right) + n^{-1} \epsilon \Phi\left(\sqrt{m(i/n)} + \frac{\log(-\epsilon/y)}{2\sqrt{m(i/n)}}\right) \right. \\ & \left. + n^{-1} \frac{1}{2\sqrt{m(i/n)}} \int_{\log \epsilon}^{\log(-x)} \phi\left(\sqrt{m(i/n)} - \frac{\log(-y)}{2\sqrt{m(i/n)}} + \frac{s}{2\sqrt{m(i/n)}}\right) e^s ds \right) (1 + o(1)) \\ = & \left(n^{-1} x \Phi\left(\sqrt{m(i/n)} + \frac{\log(x/y)}{2\sqrt{m(i/n)}}\right) + n^{-1} \epsilon \Phi\left(\sqrt{m(i/n)} + \frac{\log(-\epsilon/y)}{2\sqrt{m(i/n)}}\right) \right. \\ & \left. - n^{-1} y \Phi\left(\frac{\log(x/y)}{2\sqrt{m(i/n)}} - \sqrt{m(i/n)}\right) + n^{-1} y \Phi\left(\frac{\log(-\epsilon/y)}{2\sqrt{m(i/n)}} - \sqrt{m(i/n)}\right) \right) (1 + o(1)) \\ = & \left(n^{-1} x \Phi\left(\sqrt{m(i/n)} + \frac{\log(x/y)}{2\sqrt{m(i/n)}}\right) + n^{-1} \epsilon \Phi\left(\sqrt{m(i/n)} + \frac{\log(-\epsilon/y)}{2\sqrt{m(i/n)}}\right) \right. \\ & \left. - n^{-1} y + n^{-1} y \Phi\left(\frac{\log(y/x)}{2\sqrt{m(i/n)}} + \sqrt{m(i/n)}\right) + n^{-1} y \Phi\left(\frac{\log(-\epsilon/y)}{2\sqrt{m(i/n)}} - \sqrt{m(i/n)}\right) \right) (1 + o(1)), \end{aligned} \quad (5.3)$$

where $\phi(s) = \Phi'(s)$. Hence, it follows from (5.1) and (5.3) that

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 - \mathbb{P}(\Phi^-(F_1(X_i)) \leq \Phi^-(1 + \frac{x}{n}), \Phi^-(F_2(Y_i)) \leq \Phi^-(1 + \frac{y}{n})) \right) \\ = & -x \int_0^1 \Phi\left(\sqrt{m(s)} + \frac{\log(x/y)}{2\sqrt{m(s)}}\right) ds - y \int_0^1 \Phi\left(\sqrt{m(s)} + \frac{\log(y/x)}{2\sqrt{m(s)}}\right) ds \end{aligned}$$

for any $x < 0$ and $y < 0$, which implies that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P}\left(n(\max_{1 \leq i \leq n} F_1(X_i) - 1) \leq x, n(\max_{1 \leq i \leq n} F_2(Y_i) - 1) \leq y\right) \\ = & \lim_{n \rightarrow \infty} \prod_{i=1}^n \mathbb{P}\left(\Phi^-(F_1(X_i)) \leq \Phi^-(1 + \frac{x}{n}), \Phi^-(F_2(Y_i)) \leq \Phi^-(1 + \frac{y}{n})\right) \\ = & \exp\left(\lim_{n \rightarrow \infty} \sum_{i=1}^n \log \mathbb{P}\left(\Phi^-(F_1(X_i)) \leq \Phi^-(1 + \frac{x}{n}), \Phi^-(F_2(Y_i)) \leq \Phi^-(1 + \frac{y}{n})\right)\right) \\ = & \exp\left(-\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 - \mathbb{P}\left(\Phi^-(F_1(X_i)) \leq \Phi^-(1 + \frac{x}{n}), \Phi^-(F_2(Y_i)) \leq \Phi^-(1 + \frac{y}{n})\right)\right)\right) \\ = & \exp\left(x \int_0^1 \Phi\left(\sqrt{m(s)} + \frac{\log(x/y)}{2\sqrt{m(s)}}\right) ds + y \int_0^1 \Phi\left(\sqrt{m(s)} + \frac{\log(y/x)}{2\sqrt{m(s)}}\right) ds\right) \end{aligned}$$

for all $x < 0$ and $y < 0$. The rest for computing the tail dependence function and tail coefficient is straightforward. \square

Proof of Theorem 2. Put $U_i = F_1(X_i)$, $V_i = F_2(Y_i)$, $\hat{U}_n(u) = \frac{1}{n+1} \sum_{i=1}^n I(U_i \leq u)$, $\hat{V}_n(v) = \frac{1}{n+1} \sum_{i=1}^n I(V_i \leq v)$, $Z_i = (U_i - \frac{1}{2})(V_i - \frac{1}{2})$ and $\hat{Z}_i = \left(\hat{U}_n(U_i) - \frac{1}{2}\right)\left(\hat{V}_n(V_i) - \frac{1}{2}\right)$ for $i = 1, \dots, n$. Then

$$\left\{ \left(\hat{F}_1(X_i) - \frac{1}{2} \right) \left(\hat{F}_2(Y_i) - \frac{1}{2} \right) \right\}_{i=1}^n \stackrel{d}{=} \{\hat{Z}_i\}_{i=1}^n. \quad (5.4)$$

It is also known that

$$\sup_{0 < u < 1} \left| \frac{\sqrt{n}\{\hat{U}_n(u) - u\}}{u^\delta(1-u)^\delta} \right| = O_p(1) \quad \text{and} \quad \sup_{0 < v < 1} \left| \frac{\sqrt{n}\{\hat{V}_n(v) - v\}}{v^\delta(1-v)^\delta} \right| = O_p(1); \quad (5.5)$$

see Inequality 1 in Page 134 of Shorack and Wellner [17](1986).

Put

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{n}} \sum_{i=1}^n (\hat{U}_n(U_i) - U_i)(\hat{V}_n(V_i) - V_i), \\ I_2 &= \frac{1}{(n+1)\sqrt{n}} \sum_{i=1}^n \sum_{j=1}^n \left((I(U_j \leq U_i) - U_i)(V_i - \frac{1}{2}) \right. \\ &\quad \left. - \int_0^1 \int_0^1 (I(U_j \leq u) - u)(v - \frac{1}{2})c(u, v; \rho_i) du dv \right), \\ I_3 &= \frac{1}{(n+1)\sqrt{n}} \sum_{i=1}^n \sum_{j=1}^n \left((I(V_j \leq V_i) - V_i)(U_i - \frac{1}{2}) \right. \\ &\quad \left. - \int_0^1 \int_0^1 (I(V_j \leq v) - v)(u - \frac{1}{2})c(u, v; \rho_i) du dv \right) \end{aligned}$$

and

$$\begin{aligned} \tilde{Z}_i &= \frac{1}{(n+1)} \sum_{j=1}^n \int_0^1 \int_0^1 (I(U_i \leq u) - u)(v - \frac{1}{2})c(u, v; \rho_j) du dv \\ &\quad + \frac{1}{(n+1)} \sum_{j=1}^n \int_0^1 \int_0^1 (I(V_i \leq v) - v)(u - \frac{1}{2})c(u, v; \rho_j) du dv \\ &\quad + \left(Z_i - \frac{1}{2\pi} \arcsin\left(\frac{\rho_i}{2}\right) \right) \\ &= \tilde{Z}_{i,1} + \tilde{Z}_{i,2} + \tilde{Z}_{i,3}. \end{aligned}$$

Therefore

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\hat{Z}_i - \frac{1}{2\pi} \arcsin\left(\frac{\rho_i}{2}\right) \right) = I_1 + I_2 + I_3 + \frac{1}{\sqrt{n}} \sum_{i=1}^n \tilde{Z}_i. \quad (5.6)$$

It follows from (5.5) that

$$I_1 = O_p\left(\frac{1}{\sqrt{n}}\right). \quad (5.7)$$

Direct calculations show that $E I_2^2 = O(\frac{1}{n})$ and $E I_3^2 = O(\frac{1}{n})$, which imply that

$$I_2 = O_p\left(\frac{1}{\sqrt{n}}\right) \quad \text{and} \quad I_3 = O_p\left(\frac{1}{\sqrt{n}}\right). \quad (5.8)$$

By (5.4), (5.6)–(5.8), we have

$$\frac{1}{\sqrt{n}} l_{n1}(\alpha, \beta, \gamma) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \tilde{Z}_i + O_p(1/\sqrt{n}). \quad (5.9)$$

Using

$$\begin{cases} \frac{\partial}{\partial u} C(u, v; \rho_i) = \Phi\left(\frac{\Phi^-(v) - \rho_i \Phi^-(u)}{\sqrt{1 - \rho_i^2}}\right) := C_1(u, v; \rho_i) \\ \frac{\partial}{\partial v} C(u, v; \rho_i) = \Phi\left(\frac{\Phi^-(u) - \rho_i \Phi^-(v)}{\sqrt{1 - \rho_i^2}}\right) := C_2(u, v; \rho_i), \end{cases}$$

we have

$$\begin{aligned}
& \int_0^1 (v - \frac{1}{2}) c(u, v; \rho_i) dv \\
&= \int_0^1 (v - \frac{1}{2}) C_1(u, dv; \rho_i) \\
&= \frac{1}{2} C_1(u, 1; \rho_i) + \frac{1}{2} C_1(u, 0; \rho_i) - \int_0^1 C_1(u, v; \rho_i) dv \\
&= u - \frac{1}{2} - \int_0^u \Phi \left(\frac{\Phi^-(v) - \Phi^-(u) + (1 - \rho_i) \Phi^-(u)}{\sqrt{1 - \rho_i^2}} \right) dv \\
&\quad + \int_u^1 \left(1 - \Phi \left(\frac{\Phi^-(v) - \Phi^-(u) + (1 - \rho_i) \Phi^-(u)}{\sqrt{1 - \rho_i^2}} \right) \right) dv \\
&= u - \frac{1}{2} - \int_{-\infty}^0 \Phi \left(v + \frac{1 - \rho_i}{\sqrt{1 - \rho_i^2}} \Phi^-(u) \right) \phi \left(v \sqrt{1 - \rho_i^2} + \Phi^-(u) \right) \sqrt{1 - \rho_i^2} dv \\
&\quad + \int_0^\infty \left(1 - \Phi \left(v + \frac{1 - \rho_i}{\sqrt{1 - \rho_i^2}} \Phi^-(u) \right) \right) \phi \left(v \sqrt{1 - \rho_i^2} + \Phi^-(u) \right) \sqrt{1 - \rho_i^2} dv \\
&= u - \frac{1}{2} - \sqrt{1 - \rho_i^2} \left(\int_{-\infty}^0 \Phi(v) \phi(\Phi^-(u)) dv + O(\sqrt{1 - \rho_i}) \right) \\
&\quad + \sqrt{1 - \rho_i^2} \left(\int_0^\infty (1 - \Phi(v)) \phi(\Phi^-(u)) dv + O(\sqrt{1 - \rho_i}) \right) \\
&= u - \frac{1}{2} - \sqrt{1 - \rho_i^2} \left(\frac{1}{\sqrt{2\pi}} \phi(\Phi^-(u)) + O(\sqrt{1 - \rho_i}) \right) \\
&\quad + \sqrt{1 - \rho_i^2} \left(\frac{1}{\sqrt{2\pi}} \phi(\Phi^-(u)) + O(\sqrt{1 - \rho_i}) \right) \\
&= u - \frac{1}{2} + O(1/\log n)
\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}
& \int_0^1 (v - \frac{1}{2})^2 c(u, v; \rho_i) dv \\
&= \frac{1}{4} - 2 \int_0^1 (v - \frac{1}{2}) C_1(u, v; \rho_i) dv \\
&= (u - \frac{1}{2})^2 - 2 \int_0^u (v - \frac{1}{2}) \Phi \left(\frac{\Phi^-(v) - \Phi^-(u) + (1 - \rho_i) \Phi^-(u)}{\sqrt{1 - \rho_i^2}} \right) dv \\
&\quad + 2 \int_u^1 (v - \frac{1}{2}) \left(1 - \Phi \left(\frac{\Phi^-(v) - \Phi^-(u) + (1 - \rho_i) \Phi^-(u)}{\sqrt{1 - \rho_i^2}} \right) \right) dv \\
&= (u - \frac{1}{2})^2 - 2 \sqrt{1 - \rho_i^2} \left(\int_{-\infty}^0 (v - \frac{1}{2}) \Phi(v) \phi(\Phi^-(u)) dv + O(\sqrt{1 - \rho_i}) \right) \\
&\quad + 2 \sqrt{1 - \rho_i^2} \left(\int_0^\infty (v - \frac{1}{2}) (1 - \Phi(v)) \phi(\Phi^-(u)) dv + O(\sqrt{1 - \rho_i}) \right) \\
&= (u - \frac{1}{2})^2 - 2 \sqrt{1 - \rho_i^2} \left(-(\frac{1}{2\sqrt{2\pi}} + \frac{1}{4}) \phi(\Phi^-(u)) + O(\sqrt{1 - \rho_i}) \right) \\
&\quad + 2 \sqrt{1 - \rho_i^2} \left((\frac{1}{4} - \frac{1}{2\sqrt{2\pi}}) \phi(\Phi^-(u)) + O(\sqrt{1 - \rho_i}) \right) \\
&= (u - \frac{1}{2})^2 + \phi(\Phi^-(u)) \sqrt{1 - \rho_i^2} + O(1/\log n).
\end{aligned} \tag{5.11}$$

By (5.10), (5.11), $C(u, v; 1) = u \wedge v$ and

$$\frac{d}{d\rho} C(u, v; \rho) = \frac{1}{2\pi\sqrt{1 - \rho^2}} \exp \left(-\frac{(\Phi^-(u))^2 - 2\rho\Phi^-(u)\Phi^-(v) + (\Phi^-(v))^2}{2(1 - \rho^2)} \right) \tag{5.12}$$

(see Plackett (1954)), we have

$$\begin{aligned}
& E \tilde{Z}_{i,1}^2 = E \tilde{Z}_{i,2}^2 \\
&= \frac{1}{(n+1)^2} \sum_{j=1}^n \sum_{k=1}^n \int_0^1 \int_0^1 \int_0^1 \int_0^1 (u_1 \wedge u_2 - u_1 u_2) (v_1 - \frac{1}{2}) (v_2 - \frac{1}{2}) \times \\
&\quad c(u_1, v_1; \rho_j) c(u_2, v_2; \rho_k) dv_1 dv_2 du_1 du_2 \\
&= \int_0^1 \int_0^1 (u_1 \wedge u_2 - u_1 u_2) (u_1 - \frac{1}{2}) (u_2 - \frac{1}{2}) du_1 du_2 + O(1/\log n) \\
&= \frac{1}{720} + O(1/\log n),
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
& \mathbb{E} \tilde{Z}_{i,3}^2 \\
&= \int_0^1 \int_0^1 (u - \frac{1}{2})^2 (v - \frac{1}{2})^2 c(u, v; \rho_i) dv du - \left(\frac{1}{2\pi} \arcsin(\frac{\rho_i}{2}) \right)^2 \\
&= \int_0^1 (u - \frac{1}{2})^2 \left((u - \frac{1}{2})^2 + \phi(\Phi^-(u)) \sqrt{1 - \rho_i^2} \right) du + O(1/\log n) - \left(\frac{1}{2\pi} \arcsin(\frac{1}{2}) \right)^2 \\
&= \frac{1}{80} + \frac{\sqrt{2}\sqrt{m(i/n)}}{\sqrt{\log n}} \int_0^1 (u - \frac{1}{2})^2 \phi(\Phi^-(u)) du - \frac{1}{144} + O(1/\log n) \\
&= \frac{1}{180} + \frac{\sqrt{2}\sqrt{m(i/n)}}{\sqrt{\log n}} \int_0^1 (u - \frac{1}{2})^2 \phi(\Phi^-(u)) du + O(1/\log n),
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
& \mathbb{E}(\tilde{Z}_{i,1} \tilde{Z}_{i,2}) \\
&= \frac{1}{(n+1)^2} \sum_{j=1}^n \sum_{k=1}^n \int_0^1 \int_0^1 \int_0^1 \int_0^1 (C(u_1, v_2; \rho_i) - u_1 v_2) (v_1 - \frac{1}{2})(u_2 - \frac{1}{2}) \times \\
&\quad c(u_1, v_1; \rho_j) c(u_2, v_2; \rho_k) dv_1 du_2 du_1 dv_2 \\
&= \int_0^1 \int_0^1 (u_1 \wedge v_2 - u_1 v_2) (u_1 - \frac{1}{2})(v_2 - \frac{1}{2}) du_1 dv_2 + O(1/\log n) \\
&= \frac{1}{720} + O(1/\log n)
\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}
& \mathbb{E}(\tilde{Z}_{i,1} \tilde{Z}_{i,3}) = \mathbb{E}(\tilde{Z}_{i,2} \tilde{Z}_{i,3}) \\
&= \frac{1}{n+1} \sum_{j=1}^n \int_0^1 \int_0^1 \int_0^1 \int_0^1 (I(u_2 \leq u_1) - u_1) (v_1 - \frac{1}{2})(u_2 - \frac{1}{2})(v_2 - \frac{1}{2}) \times \\
&\quad c(u_1, v_1; \rho_j) c(u_2, v_2; \rho_i) dv_1 dv_2 du_1 du_2 \\
&= \int_0^1 \int_0^1 (I(u_2 \leq u_1) - u_1) (u_2 - \frac{1}{2})(u_1 - \frac{1}{2})(u_2 - \frac{1}{2}) du_1 du_2 + O(1/\log n) \\
&= -\frac{1}{360} + O(1/\log n).
\end{aligned} \tag{5.16}$$

Hence, it follows from (5.13)–(5.16) that

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E} \tilde{Z}_i^2 = \frac{\sqrt{2} \int_0^1 \sqrt{\alpha + \beta s^\gamma} ds}{\sqrt{\log n}} \int_0^1 \left(u - \frac{1}{2} \right)^2 \phi(\Phi^-(u)) du + O(1/\log n). \tag{5.17}$$

It is easy to check that

$$\mathbb{E} \left(\frac{1}{n} \sum_{i=1}^n (\tilde{Z}_i^2 - \mathbb{E} \tilde{Z}_i^2) \right)^2 = \frac{1}{n^2} \sum_{i=1}^n \mathbb{E} (\tilde{Z}_i^2 - \mathbb{E} \tilde{Z}_i^2)^2 = O(1/n),$$

which, combining with (5.17), implies that

$$\sum_{i=1}^n \left(\frac{(\log n)^{1/4}}{\sqrt{n}} \tilde{Z}_i \right)^2 \xrightarrow{p} \sqrt{2} \left(\int_0^1 \sqrt{\alpha + \beta s^\gamma} ds \right) \left(\int_0^1 (u - \frac{1}{2})^2 \phi(\Phi^-(u)) du \right). \tag{5.18}$$

Obviously we have

$$\max_{1 \leq i \leq n} \left| \frac{(\log n)^{1/4}}{\sqrt{n}} \tilde{Z}_i \right| \xrightarrow{p} 0 \quad \text{and} \quad \mathbb{E} \left(\max_{1 \leq i \leq n} \frac{(\log n)^{1/2}}{n} \tilde{Z}_i^2 \right) = o(1). \tag{5.19}$$

Hence, it follows from (5.9), (5.18), (5.19) and Theorem 3.2 of Hall and Heyde (1980) that

$$\frac{(\log n)^{1/4}}{\sqrt{n}} l_{n1}(\alpha, \beta, \gamma) \rightarrow N \left(0, \sqrt{2} \left(\int_0^1 \sqrt{\alpha + \beta s^\gamma} ds \right) \left(\int_0^1 (u - \frac{1}{2})^2 \phi(\Phi^-(u)) du \right) \right). \tag{5.20}$$

Note that the above limit has a nonstandard rate, which can be explained as follows. When $\rho_i = 1$ for $i = 1, \dots, n$, we have

$$l_{n1}(\alpha, \beta, \gamma) = \sum_{i=1}^n \left(\left(\hat{U}_n(U_i) - \frac{1}{2} \right)^2 - \frac{1}{12} \right) = \sum_{i=1}^n \left(\left(\frac{i}{n+1} - \frac{1}{2} \right)^2 - \frac{1}{12} \right),$$

which becomes a constant. However, $l_{n2}(\alpha, \beta, \gamma)$ and $l_{n3}(\alpha, \beta, \gamma)$ are non-degenerate due to the involved factors $(i/n)^\gamma$ and $(i/n)^\gamma \log(i/n)$. That is, deriving the asymptotic limit of $l_{n1}(\alpha, \beta, \gamma)$ needs finer expansions than the other two quantities. Below we show the asymptotic limits for both l_{n2} and l_{n3} have the standard rate $1/\sqrt{n}$.

Define

$$\begin{aligned}\tilde{Z}_i^* &= \frac{1}{(n+1)} \sum_{j=1}^n \left(\frac{j}{n}\right)^\gamma \int_0^1 \int_0^1 (I(U_i \leq u) - u)(v - \frac{1}{2}) c(u, v; \rho_j) du dv \\ &\quad + \frac{1}{(n+1)} \sum_{j=1}^n \left(\frac{j}{n}\right)^\gamma \int_0^1 \int_0^1 (I(V_i \leq v) - v)(u - \frac{1}{2}) c(u, v; \rho_j) du dv \\ &\quad + \left(Z_i - \frac{1}{2\pi} \arcsin\left(\frac{\rho_i}{2}\right)\right) \left(\frac{i}{n}\right)^\gamma \\ &= \tilde{Z}_{i,1}^* + \tilde{Z}_{i,2}^* + \tilde{Z}_{i,3}^*\end{aligned}$$

and

$$\begin{aligned}\tilde{Z}_i^{**} &= \frac{1}{(n+1)} \sum_{j=1}^n \left(\frac{j}{n}\right)^\gamma \log\left(\frac{j}{n}\right) \int_0^1 \int_0^1 (I(U_i \leq u) - u)(v - \frac{1}{2}) c(u, v; \rho_j) du dv \\ &\quad + \frac{1}{(n+1)} \sum_{j=1}^n \left(\frac{j}{n}\right)^\gamma \log\left(\frac{j}{n}\right) \int_0^1 \int_0^1 (I(V_i \leq v) - v)(u - \frac{1}{2}) c(u, v; \rho_j) du dv \\ &\quad + \left(Z_i - \frac{1}{2\pi} \arcsin\left(\frac{\rho_i}{2}\right)\right) \left(\frac{i}{n}\right)^\gamma \log\left(\frac{i}{n}\right) \\ &= \tilde{Z}_{i,1}^{**} + \tilde{Z}_{i,2}^{**} + \tilde{Z}_{i,3}^{**}.\end{aligned}$$

Similar to the proof of (5.9), we can show that

$$\begin{cases} \frac{1}{\sqrt{n}} l_{n2}(\alpha, \beta, \gamma) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \tilde{Z}_i^* + O_p(1/\sqrt{n}) \\ \frac{1}{\sqrt{n}} l_{n3}(\alpha, \beta, \gamma) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \tilde{Z}_i^{**} + O_p(1/\sqrt{n}). \end{cases} \quad (5.21)$$

Like the proofs of (5.13)–(5.16), we can show that

$$\begin{aligned}\mathbb{E} \tilde{Z}_{i,1}^{*2} &= \mathbb{E} \tilde{Z}_{i,2}^{*2} \\ &= \left(\frac{1}{n} \sum_{j=1}^n \left(\frac{j}{n}\right)^\gamma\right)^2 \int_0^1 \int_0^1 (u_1 \wedge u_2 - u_1 u_2) (u_1 - \frac{1}{2})(u_2 - \frac{1}{2}) du_1 du_2 + O(1/\log n) \\ &= \frac{1}{720(1+\gamma)^2} + O(1/\log n),\end{aligned}$$

$$\begin{cases} \mathbb{E} \tilde{Z}_{i,3}^{*2} = \frac{1}{180} \left(\frac{i}{n}\right)^{2\gamma} + O(1/\sqrt{\log n}), & \mathbb{E}(\tilde{Z}_{i,1}^* \tilde{Z}_{i,2}^*) = \frac{1}{720(1+\gamma)^2} + O(1/\log n) \\ \mathbb{E}(\tilde{Z}_{i,1}^* \tilde{Z}_{i,3}^*) = \mathbb{E}(\tilde{Z}_{i,2}^* \tilde{Z}_{i,3}^*) = -\frac{1}{360(1+\gamma)} \left(\frac{i}{n}\right)^\gamma + O(1/\sqrt{\log n}), \end{cases}$$

$$\begin{aligned}\mathbb{E} \tilde{Z}_{i,1}^{**2} &= \mathbb{E} \tilde{Z}_{i,2}^{**2} \\ &= \left(\frac{1}{n} \sum_{j=1}^n \left(\frac{j}{n}\right)^\gamma \log\left(\frac{j}{n}\right)\right)^2 \int_0^1 \int_0^1 (u_1 \wedge u_2 - u_1 u_2) (u_1 - \frac{1}{2})(u_2 - \frac{1}{2}) du_1 du_2 + O(1/\log n) \\ &= \frac{1}{720(1+\gamma)^4} + O(1/\log n)\end{aligned}$$

and

$$\begin{cases} \mathbb{E} \tilde{Z}_{i,3}^{**2} = \frac{1}{180} \left(\frac{i}{n}\right)^{2\gamma} \log^2\left(\frac{i}{n}\right) + O(1/\sqrt{\log n}), & \mathbb{E}(\tilde{Z}_{i,1}^{**} \tilde{Z}_{i,2}^{**}) = \frac{1}{720(1+\gamma)^4} + O(1/\log n) \\ \mathbb{E}(\tilde{Z}_{i,1}^{**} \tilde{Z}_{i,3}^{**}) = \mathbb{E}(\tilde{Z}_{i,2}^{**} \tilde{Z}_{i,3}^{**}) = \frac{1}{360(1+\gamma)^2} \left(\frac{i}{n}\right)^\gamma \log\left(\frac{i}{n}\right) + O(1/\sqrt{\log n}), \end{cases}$$

which imply that

$$\begin{cases} \mathbb{E} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \tilde{Z}_i^*\right)^2 = \frac{1}{n} \sum_{i=1}^n \mathbb{E}(\tilde{Z}_{i,1}^* + \tilde{Z}_{i,2}^* + \tilde{Z}_{i,3}^*)^2 \rightarrow \frac{1}{180(1+2\gamma)} - \frac{1}{180(1+\gamma)^2} \\ \mathbb{E} \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \tilde{Z}_i^{**}\right)^2 = \frac{1}{n} \sum_{i=1}^n \mathbb{E}(\tilde{Z}_{i,1}^{**} + \tilde{Z}_{i,2}^{**} + \tilde{Z}_{i,3}^{**})^2 \rightarrow \frac{1}{90(1+2\gamma)^3} - \frac{1}{180(1+\gamma)^4}. \end{cases} \quad (5.22)$$

Like the proof of (5.20), by using (5.22), we can show that

$$\frac{1}{\sqrt{n}} l_{n2}(\alpha, \beta, \gamma) \xrightarrow{d} N\left(0, \frac{1}{180(1+2\gamma)} - \frac{1}{180(1+\gamma)^2}\right)$$

and

$$\frac{1}{\sqrt{n}} l_{n3}(\alpha, \beta, \gamma) \xrightarrow{d} N\left(0, \frac{1}{90(1+2\gamma)^3} - \frac{1}{180(1+\gamma)^4}\right).$$

Some further tedious calculations show that

$$\begin{aligned} & \frac{(\log n)^{1/4}}{n} \sum_{i=1}^n E(\tilde{Z}_i \tilde{Z}_i^*) \\ = & \frac{(\log n)^{1/4}}{n} \sum_{i=1}^n \sum_{j=1}^3 \sum_{k=1}^3 E(\tilde{Z}_{i,j} \tilde{Z}_{i,k}^*) \\ = & \frac{(\log n)^{1/4}}{n} \sum_{i=1}^n \left(\frac{1}{720(1+\gamma)} + \frac{1}{720(1+\gamma)} - \frac{1}{360} \left(\frac{i}{n}\right)^\gamma \right. \\ & \left. + \frac{1}{720(1+\gamma)} + \frac{1}{720(1+\gamma)} - \frac{1}{360} \left(\frac{i}{n}\right)^\gamma - \frac{1}{360(1+\gamma)} - \frac{1}{360(1+\gamma)} + \frac{1}{180} \left(\frac{i}{n}\right)^\gamma + O(1/\sqrt{\log n}) \right) \\ = & o(1), \\ & \frac{(\log n)^{1/4}}{n} \sum_{i=1}^n E(\tilde{Z}_i \tilde{Z}_i^{**}) \\ = & \frac{(\log n)^{1/4}}{n} \sum_{i=1}^n \sum_{j=1}^3 \sum_{k=1}^3 E(\tilde{Z}_{i,j} \tilde{Z}_{i,k}^{**}) \\ = & \frac{(\log n)^{1/4}}{n} \sum_{i=1}^n \left(-\frac{1}{720(1+\gamma)^2} - \frac{1}{720(1+\gamma)^2} - \frac{1}{360} \left(\frac{i}{n}\right)^\gamma \log\left(\frac{i}{n}\right) \right. \\ & \left. - \frac{1}{720(1+\gamma)^2} - \frac{1}{720(1+\gamma)^2} - \frac{1}{360} \left(\frac{i}{n}\right)^\gamma \log\left(\frac{i}{n}\right) + \frac{1}{360(1+\gamma)^2} + \frac{1}{360(1+\gamma)^2} \right. \\ & \left. + \frac{1}{180} \left(\frac{i}{n}\right)^\gamma \log\left(\frac{i}{n}\right) + O(1/\sqrt{\log n}) \right) \\ = & o(1) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n E(\tilde{Z}_i^* \tilde{Z}_i^{**}) \\ = & \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^3 \sum_{k=1}^3 E(\tilde{Z}_{i,j}^* \tilde{Z}_{i,k}^{**}) \\ = & \frac{1}{n} \sum_{i=1}^n \left(-\frac{1}{720(1+\gamma)^3} - \frac{1}{720(1+\gamma)^3} - \frac{1}{360(1+\gamma)} \left(\frac{i}{n}\right)^\gamma \log\left(\frac{i}{n}\right) \right. \\ & \left. - \frac{1}{720(1+\gamma)^3} - \frac{1}{720(1+\gamma)^3} - \frac{1}{360(1+\gamma)} \left(\frac{i}{n}\right)^\gamma \log\left(\frac{i}{n}\right) + \frac{1}{360(1+\gamma)^2} \left(\frac{i}{n}\right)^\gamma + \frac{1}{360(1+\gamma)^2} \left(\frac{i}{n}\right)^\gamma \right. \\ & \left. + \frac{1}{180} \left(\frac{i}{n}\right)^{2\gamma} \log\left(\frac{i}{n}\right) + O(1/\sqrt{\log n}) \right) \\ = & \frac{1}{180(1+\gamma)^3} - \frac{1}{180(1+2\gamma)^2} + o(1). \end{aligned}$$

Hence, by Cramér device, we can show that

$$\left(\frac{(\log n)^{1/4}}{\sqrt{n}} l_{n1}(\alpha, \beta, \gamma), \frac{1}{\sqrt{n}} l_{n2}(\alpha, \beta, \gamma), \frac{1}{\sqrt{n}} l_{n3}(\alpha, \beta, \gamma) \right)^T \xrightarrow{d} N(0, \Sigma). \quad (5.23)$$

It is straightforward to check that

$$\left\{ \begin{array}{l} \frac{\log n}{n} \frac{\partial l_{n1}(\alpha, \beta, \gamma)}{\partial \alpha} \rightarrow \frac{\sqrt{3}}{6\pi}, \frac{\log n}{n} \frac{\partial l_{n1}(\alpha, \beta, \gamma)}{\partial \beta} \rightarrow \frac{\sqrt{3}}{6\pi(1+\gamma)}, \\ \frac{\log n}{n} \frac{\partial l_{n1}(\alpha, \beta, \gamma)}{\partial \gamma} \rightarrow -\frac{\sqrt{3}\beta}{6\pi(1+\gamma)^2}, \frac{\log n}{n} \frac{\partial l_{n2}(\alpha, \beta, \gamma)}{\partial \alpha} \rightarrow \frac{\sqrt{3}}{6\pi(1+\gamma)}, \\ \frac{\log n}{n} \frac{\partial l_{n2}(\alpha, \beta, \gamma)}{\partial \beta} \rightarrow \frac{\sqrt{3}}{6\pi(1+2\gamma)}, \frac{\log n}{n} \frac{\partial l_{n2}(\alpha, \beta, \gamma)}{\partial \gamma} \rightarrow -\frac{\sqrt{3}\beta}{6\pi(1+2\gamma)^2}, \\ \frac{\log n}{n} \frac{\partial l_{n3}(\alpha, \beta, \gamma)}{\partial \alpha} \rightarrow -\frac{\sqrt{3}}{6\pi(1+\gamma)^2}, \frac{\log n}{n} \frac{\partial l_{n3}(\alpha, \beta, \gamma)}{\partial \beta} \rightarrow -\frac{\sqrt{3}}{6\pi(1+2\gamma)^2}, \\ \frac{\log n}{n} \frac{\partial l_{n3}(\alpha, \beta, \gamma)}{\partial \gamma} \rightarrow \frac{\sqrt{3}\beta}{3\pi(1+2\gamma)^3}. \end{array} \right. \quad (5.24)$$

Hence, the theorem follows from (5.23), (5.24) and Taylor expansion. \square

Proof of Theorem 3. As in the proof of Theorem 2, we define

$$\begin{aligned} \bar{Z}_i &= \frac{1}{(n+1)} \sum_{j=1}^n \int_0^1 \int_0^1 \frac{\Phi^-(v)}{\phi(\Phi^-(u))} (I(U_i \leq u) - u) c(u, v; \rho_j) du dv \\ &\quad + \frac{1}{(n+1)} \sum_{j=1}^n \int_0^1 \int_0^1 \frac{\Phi^-(u)}{\phi(\Phi^-(v))} (I(V_i \leq v) - v) c(u, v; \rho_j) du dv \\ &\quad + \left(\Phi^-(U_i) \Phi^-(V_i) - \rho_i \right) \\ &= \bar{Z}_{i,1} + \bar{Z}_{i,2} + \bar{Z}_{i,3}, \end{aligned}$$

$$\begin{aligned}
\bar{Z}_i^* &= \frac{1}{(n+1)} \sum_{j=1}^n \left(\frac{j}{n}\right)^\gamma \int_0^1 \int_0^1 \frac{\Phi^-(v)}{\phi(\Phi^-(u))} (I(U_i \leq u) - u) c(u, v; \rho_j) du dv \\
&\quad + \frac{1}{(n+1)} \sum_{j=1}^n \left(\frac{j}{n}\right)^\gamma \int_0^1 \int_0^1 \frac{\Phi^-(u)}{\phi(\Phi^-(v))} (I(V_i \leq v) - v) c(u, v; \rho_j) du dv \\
&\quad + \left(\Phi^-(U_i) \Phi^-(V_i) - \rho_i \right) \left(\frac{i}{n}\right)^\gamma \\
&= \bar{Z}_{i,1}^* + \bar{Z}_{i,2}^* + \bar{Z}_{i,3}^*
\end{aligned}$$

and

$$\begin{aligned}
\bar{Z}_i^{**} &= \frac{1}{(n+1)} \sum_{j=1}^n \left(\frac{j}{n}\right)^\gamma \log\left(\frac{j}{n}\right) \int_0^1 \int_0^1 \frac{\Phi^-(v)}{\phi(\Phi^-(u))} (I(U_i \leq u) - u) c(u, v; \rho_j) du dv \\
&\quad + \frac{1}{(n+1)} \sum_{j=1}^n \left(\frac{j}{n}\right)^\gamma \log\left(\frac{j}{n}\right) \int_0^1 \int_0^1 \frac{\Phi^-(u)}{\phi(\Phi^-(v))} (I(V_i \leq v) - v) c(u, v; \rho_j) du dv \\
&\quad + \left(\Phi^-(U_i) \Phi^-(V_i) - \rho_i \right) \left(\frac{i}{n}\right)^\gamma \log\left(\frac{i}{n}\right) \\
&= \bar{Z}_{i,1}^{**} + \bar{Z}_{i,2}^{**} + \bar{Z}_{i,3}^{**}.
\end{aligned}$$

Since

$$\int_0^1 \Phi^-(v) c(u, v; \rho_i) dv = \rho_i \Phi^-(u) \quad \text{and} \quad \int_0^1 \Phi^-(u) c(u, v; \rho_i) du = \rho_i \Phi^-(v),$$

we have

$$\left\{ \begin{array}{l} \bar{Z}_{i,1} = \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right) \int_0^1 \frac{\Phi^-(u)}{\phi(\Phi^-(u))} (I(U_i \leq u) - u) du, \\ \bar{Z}_{i,2} = \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right) \int_0^1 \frac{\Phi^-(v)}{\phi(\Phi^-(v))} (I(V_i \leq v) - v) dv, \\ \bar{Z}_{i,1}^* = \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \left(\frac{j}{n}\right)^\gamma \right) \int_0^1 \frac{\Phi^-(u)}{\phi(\Phi^-(u))} (I(U_i \leq u) - u) du, \\ \bar{Z}_{i,2}^* = \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \left(\frac{j}{n}\right)^\gamma \right) \int_0^1 \frac{\Phi^-(v)}{\phi(\Phi^-(v))} (I(V_i \leq v) - v) dv, \\ \bar{Z}_{i,1}^{**} = \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \left(\frac{j}{n}\right)^\gamma \log\left(\frac{j}{n}\right) \right) \int_0^1 \frac{\Phi^-(u)}{\phi(\Phi^-(u))} (I(U_i \leq u) - u) du, \\ \bar{Z}_{i,2}^{**} = \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \left(\frac{j}{n}\right)^\gamma \log\left(\frac{j}{n}\right) \right) \int_0^1 \frac{\Phi^-(v)}{\phi(\Phi^-(v))} (I(V_i \leq v) - v) dv. \end{array} \right.$$

It is straightforward to check that

$$\begin{aligned}
E \bar{Z}_{i,1}^2 &= \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right)^2 \int_0^1 \int_0^1 \frac{\Phi^-(u_1) \Phi^-(u_2)}{\phi(\Phi^-(u_1)) \phi(\Phi^-(u_2))} (u_1 \wedge u_2 - u_1 u_2) du_1 du_2 \\
&= \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right)^2 2 \int_0^1 \int_0^{u_1} \frac{\Phi^-(u_1) \Phi^-(u_2)}{\phi(\Phi^-(u_1)) \phi(\Phi^-(u_2))} u_2 (1 - u_1) du_2 du_1 \\
&= \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right)^2 \int_0^1 \frac{\Phi^-(u_1)}{\phi(\Phi^-(u_1))} (1 - u_1) \int_{-\infty}^{\Phi^-(u_1)} \Phi(u_2) du_2^2 du_1 \\
&= \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right)^2 \int_0^1 \frac{\Phi^-(u_1)}{\phi(\Phi^-(u_1))} (1 - u_1) \{ (\Phi^-(u_1))^2 u_1 + \int_{-\infty}^{\Phi^-(u_1)} u_2 d\phi(u_2) \} du_1 \\
&= \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right)^2 \int_0^1 \frac{\Phi^-(u_1)}{\phi(\Phi^-(u_1))} (1 - u_1) \{ (\Phi^-(u_1))^2 u_1 + \Phi^-(u_1) \phi(\Phi^-(u_1)) - u_1 \} du_1 \quad (5.25) \\
&= \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right)^2 \int_{-\infty}^{\infty} u (1 - \Phi(u)) \{ u^2 \Phi(u) + u \phi(u) - \Phi(u) \} du \\
&= - \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right)^2 \int_{-\infty}^{\infty} u (1 - \Phi(u)) d\phi(u) \\
&= \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right)^2 \int_{-\infty}^{\infty} \phi(u) \{ 1 - \Phi(u) - u \phi(u) \} du \\
&= \frac{1}{2} \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right)^2
\end{aligned}$$

by noting that $u(1 - \Phi(u))\Phi(u)$, $u^3(1 - \Phi(u))\Phi(u)$ and $u\phi^2(u)$ are odd functions,

$$E \bar{Z}_{i,2}^2 = \frac{1}{2} \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right)^2 \quad \text{and} \quad E \bar{Z}_{i,3}^2 = 1 + 2\rho_i^2 - \rho_i^2 = 1 + \rho_i^2. \quad (5.26)$$

By (5.12), we have

$$\int_0^1 \int_0^1 \frac{\Phi^-(u) \Phi^-(v)}{\phi(\Phi^-(u)) \phi(\Phi^-(v))} \frac{dC(u, v; \rho)}{d\rho} du dv = \rho$$

for any $\rho \in (-1, 1)$. Taking derivative with respect to ρ at both sides, we have

$$\int_0^1 \int_0^1 \frac{\Phi^-(u)\Phi^-(v)}{\phi(\Phi^-(u))\phi(\Phi^-(v))} \frac{d^2 C(u, v; \rho)}{d\rho^2} dudv = 1$$

for any $\rho \in (-1, 1)$. Therefore

$$\begin{aligned} \frac{1}{2} &= \int_0^1 \int_0^1 \frac{\Phi^-(u)\Phi^-(v)}{\phi(\Phi^-(u))\phi(\Phi^-(v))} (u \wedge v - uv) dudv \\ &= \int_0^1 \int_0^1 \frac{\Phi^-(u)\Phi^-(v)}{\phi(\Phi^-(u))\phi(\Phi^-(v))} (C(u, v; 1) - uv) dudv \\ &= \int_0^1 \int_0^1 \frac{\Phi^-(u)\Phi^-(v)}{\phi(\Phi^-(u))\phi(\Phi^-(v))} (C(u, v; \rho_i) - uv) dudv \\ &\quad + \rho_i(1 - \rho_i) + \frac{1}{2}(1 - \rho_i)^2 + o(1/\log^2 n), \end{aligned}$$

which gives

$$\begin{aligned} E(\bar{Z}_{i,1}\bar{Z}_{i,2}) &= \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right)^2 \int_0^1 \int_0^1 \frac{\Phi^-(u)\Phi^-(v)}{\phi(\Phi^-(u))\phi(\Phi^-(v))} (C(u, v; \rho_i) - uv) dudv \\ &= \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right)^2 \left(\frac{1}{2} - \rho_i(1 - \rho_i) - \frac{1}{2}(1 - \rho_i)^2 \right) + o(1/\log^2 n) \\ &= \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right)^2 \frac{\rho_i^2}{2} + o(1/\log^2 n). \end{aligned} \tag{5.27}$$

Since

$$\begin{aligned} E(I(U_i \leq u)\Phi^-(U_i)\Phi^-(V_i)) &= E\left(I(U_i \leq u)\Phi^-(U_i) E(\Phi^-(V_i)|\Phi^-(U_i))\right) \\ &= E(I(U_i \leq u)\Phi^-(U_i)\rho_i\Phi^-(U_i)) \\ &= \rho_i \int_{-\infty}^{\Phi^-(u)} v^2 \phi(v) dv \\ &= -\rho_i \int_{-\infty}^{\Phi^-(u)} v d\phi(v) \\ &= -\rho_i (\Phi^-(u)\phi(\Phi^-(u)) - u), \end{aligned}$$

we have

$$\begin{aligned} E(\bar{Z}_{i,1}\bar{Z}_{i,3}) &= E(\bar{Z}_{i,2}\bar{Z}_{i,3}) \\ &= \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right) \int_0^1 \frac{\Phi^-(u)}{\phi(\Phi^-(u))} \rho_i (-\Phi^-(u)\phi(\Phi^-(u)) + u - u) du \\ &= -\left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right) \rho_i. \end{aligned} \tag{5.28}$$

Put $\bar{\rho} = n^{-1} \sum_{j=1}^n \rho_j$. Then

$$\begin{cases} \bar{\rho} = 1 - \frac{\alpha+\beta/(1+\gamma)}{\log n} + o(1/\log^2 n) \\ \frac{1}{n} \sum_{j=1}^n \rho_j^2 = \frac{1}{n} \sum_{j=1}^n (\rho_j - \bar{\rho})^2 + \bar{\rho}^2 = \frac{\beta^2}{\log^2 n} \left(\frac{1}{1+2\gamma} - \frac{1}{(1+\gamma)^2} \right) + \bar{\rho}^2 + o(1/\log^2 n). \end{cases} \tag{5.29}$$

It follows from (5.25)–(5.29) that

$$\begin{aligned} &\frac{1}{n} \sum_{i=1}^n E \bar{Z}_i^2 \\ &= \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right)^2 + 1 + \frac{1}{n} \sum_{i=1}^n \rho_i^2 + \frac{1}{n} \sum_{i=1}^n \rho_i^2 \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right)^2 \\ &\quad - \frac{4}{n} \sum_{i=1}^n \rho_i \left(\frac{1}{n+1} \sum_{j=1}^n \rho_j \right) + o(1/\log^2 n) \\ &= \bar{\rho}^2 + 1 + \bar{\rho}^2 + \frac{1}{n} \sum_{j=1}^n (\rho_j - \bar{\rho})^2 + (\bar{\rho}^2 + \frac{1}{n} \sum_{j=1}^n (\rho_j - \bar{\rho})^2) \bar{\rho}^2 \\ &\quad - 4\bar{\rho}^2 + o(1/\log^2 n) \\ &= 4 \left(\frac{\alpha+\beta/(1+\gamma)}{\log n} \right)^2 + \frac{2\beta^2}{\log^2 n} \left(\frac{1}{1+2\gamma} - \frac{1}{(1+\gamma)^2} \right) + o(1/\log^2 n). \end{aligned} \tag{5.30}$$

Similarly we can show that

$$\begin{aligned} &\frac{1}{n} \sum_{i=1}^n E \bar{Z}_i^{*2} \\ &= \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^3 \sum_{k=1}^3 E(\bar{Z}_{i,j}^* \bar{Z}_{i,k}^*) \\ &= \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{2(1+\gamma)^2} + \frac{1}{2(1+\gamma)^2} - \frac{1}{1+\gamma} \left(\frac{i}{n} \right)^\gamma \right. \\ &\quad \left. + \frac{1}{2(1+\gamma)^2} + \frac{1}{2(1+\gamma)^2} - \frac{1}{1+\gamma} \left(\frac{i}{n} \right)^\gamma - \frac{1}{1+\gamma} \left(\frac{i}{n} \right)^\gamma - \frac{1}{1+\gamma} \left(\frac{i}{n} \right)^\gamma + 2 \left(\frac{i}{n} \right)^{2\gamma} \right) + o(1) \\ &= \frac{2}{1+2\gamma} - \frac{2}{(1+\gamma)^2} + o(1), \end{aligned} \tag{5.31}$$

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \mathbb{E} \bar{Z}_i^{**2} \\
&= \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^3 \sum_{k=1}^3 \mathbb{E}(\bar{Z}_{i,j}^{**} \bar{Z}_{i,k}^{**}) \\
&= \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{2(1+\gamma)^4} + \frac{1}{2(1+\gamma)^4} + \frac{1}{(1+\gamma)^2} \left(\frac{i}{n}\right)^\gamma \log\left(\frac{i}{n}\right) \right. \\
&\quad \left. + \frac{1}{2(1+\gamma)^4} + \frac{1}{2(1+\gamma)^4} + \frac{1}{(1+\gamma)^2} \left(\frac{i}{n}\right)^\gamma \log\left(\frac{i}{n}\right) \right. \\
&\quad \left. + \frac{1}{(1+\gamma)^2} \left(\frac{i}{n}\right)^\gamma \log\left(\frac{i}{n}\right) + \frac{1}{(1+\gamma)^2} \left(\frac{i}{n}\right)^\gamma \log\left(\frac{i}{n}\right) + 2\left(\frac{i}{n}\right)^{2\gamma} \log^2\left(\frac{i}{n}\right) \right) + o(1) \\
&= \frac{4}{(1+2\gamma)^3} - \frac{2}{(1+\gamma)^4} + o(1),
\end{aligned} \tag{5.32}$$

$$\frac{\log n}{n} \sum_{i=1}^n \mathbb{E}(\bar{Z}_i \bar{Z}_i^*) = o(1), \quad \frac{\log n}{n} \sum_{i=1}^n \mathbb{E}(\bar{Z}_i \bar{Z}_i^{**}) = o(1) \tag{5.33}$$

and

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \mathbb{E}(\bar{Z}_i^* \bar{Z}_i^{**}) \\
&= \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^3 \sum_{k=1}^3 \mathbb{E}(\bar{Z}_{i,j}^* \bar{Z}_{i,k}^{**}) \\
&= \frac{1}{n} \sum_{i=1}^n \left(-\frac{1}{2(1+\gamma)^3} - \frac{1}{2(1+\gamma)^3} - \frac{1}{1+\gamma} \left(\frac{i}{n}\right)^\gamma \log\left(\frac{i}{n}\right) \right. \\
&\quad \left. - \frac{1}{2(1+\gamma)^3} - \frac{1}{2(1+\gamma)^3} - \frac{1}{1+\gamma} \left(\frac{i}{n}\right)^\gamma \log\left(\frac{i}{n}\right) \right. \\
&\quad \left. + \frac{1}{(1+\gamma)^2} \left(\frac{i}{n}\right)^\gamma + \frac{1}{(1+\gamma)^2} \left(\frac{i}{n}\right)^\gamma + 2\left(\frac{i}{n}\right)^{2\gamma} \log\left(\frac{i}{n}\right) \right) + o(1) \\
&= -\frac{2}{(1+2\gamma)^2} + \frac{2}{(1+\gamma)^3} + o(1).
\end{aligned} \tag{5.34}$$

Therefore, using (5.30)–(5.34) and the same arguments in proving (5.23), we can show that

$$\left(\frac{\log n}{\sqrt{n}} l_{n1}^*(\alpha, \beta, \gamma), \frac{1}{\sqrt{n}} l_{n2}^*(\alpha, \beta, \gamma), \frac{1}{\sqrt{n}} l_{n3}^*(\alpha, \beta, \gamma) \right)^T \xrightarrow{d} N(0, \Sigma^*). \tag{5.35}$$

It is straightforward to check that

$$\left\{ \begin{array}{l} \frac{\log n}{n} \frac{\partial l_{n1}^*(\alpha, \beta, \gamma)}{\partial \alpha} = 1, \frac{\log n}{n} \frac{\partial l_{n1}^*(\alpha, \beta, \gamma)}{\partial \beta} \rightarrow \frac{1}{1+\gamma}, \frac{\log n}{n} \frac{\partial l_{n1}^*(\alpha, \beta, \gamma)}{\partial \gamma} \rightarrow -\frac{\beta}{(1+\gamma)^2}, \\ \frac{\log n}{n} \frac{\partial l_{n2}^*(\alpha, \beta, \gamma)}{\partial \alpha} \rightarrow \frac{1}{1+\gamma}, \frac{\log n}{n} \frac{\partial l_{n2}^*(\alpha, \beta, \gamma)}{\partial \beta} \rightarrow \frac{1}{1+2\gamma}, \frac{\log n}{n} \frac{\partial l_{n2}^*(\alpha, \beta, \gamma)}{\partial \gamma} \rightarrow -\frac{\beta}{(1+2\gamma)^2}, \\ \frac{\log n}{n} \frac{\partial l_{n3}^*(\alpha, \beta, \gamma)}{\partial \alpha} \rightarrow -\frac{1}{(1+\gamma)^2}, \frac{\log n}{n} \frac{\partial l_{n3}^*(\alpha, \beta, \gamma)}{\partial \beta} \rightarrow -\frac{1}{(1+2\gamma)^2}, \frac{\log n}{n} \frac{\partial l_{n3}^*(\alpha, \beta, \gamma)}{\partial \gamma} \rightarrow \frac{2\beta}{(1+2\gamma)^3}, \end{array} \right. \tag{5.36}$$

Hence, the theorem follows from (5.35), (5.36) and Taylor expansions. \square

Proof of Theorem 4. It follows from the proof of Theorem 2 with known $\gamma = 1$. \square

Proof of Theorem 5. It follows from the proof of Theorem 3 with known $\gamma = 1$. \square

Proof of Theorem 6. Note that

$$(\log n) Q''(s) \rightarrow -\frac{\sqrt{3} m''(s)}{6\pi} \quad \text{and} \quad \cos(2\pi Q(s)) \rightarrow \frac{\sqrt{3}}{2} \tag{5.37}$$

and

$$\begin{aligned}
& \mathbb{E} \left((F_1(X_i) - \tfrac{1}{2})(F_2(Y_i) - \tfrac{1}{2}) - \tfrac{1}{2\pi} \arcsin(\tfrac{\rho_i}{2}) \right)^2 \\
&= \mathbb{E} \left((F_1(X_i) - \tfrac{1}{2})^2 (F_2(Y_i) - \tfrac{1}{2})^2 - \left(\tfrac{1}{2\pi} \arcsin(\tfrac{\rho_i}{2}) \right)^2 \right) \\
&\rightarrow \tfrac{1}{80} - \left(\tfrac{1}{12} \right)^2 = \tfrac{1}{180}.
\end{aligned} \tag{5.38}$$

It follows from (5.38) and the standard arguments in local linear estimation (e.g., Fan and Gijbels [3]) that

$$\sqrt{nh} \left(\hat{Q}(s) - Q(s) - \frac{1}{2} Q''(s) h^2 \int_{-1}^1 t^2 k(t) dt \right) \xrightarrow{d} N \left(0, \frac{1}{180} \int_{-1}^1 k^2(t) dt \right). \tag{5.39}$$

Hence it follows from (5.37) and (5.39) that

$$\begin{aligned} & \frac{\sqrt{nh}}{\log n}(\hat{m}(s) - m(s)) \\ &= -4\pi \cos(2\pi Q(s))\sqrt{nh}(\hat{Q}(s) - Q(s)) + o_p(1) \\ &\xrightarrow{d} N\left(\frac{1}{2}\lambda m''(s) \int_{-1}^1 t^2 k(t) dt, \frac{\pi^2}{15} \int_{-1}^1 k^2(t) dt\right), \end{aligned}$$

i.e., the theorem holds. □

Proof of Theorem 7. It follows from standard arguments in local linear estimation. □

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